

INNER FUNCTIONS ON THE BIDISK AND ASSOCIATED HILBERT SPACES

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ABSTRACT. Matrix valued inner functions on the bidisk have a number of natural subspaces of the Hardy space on the torus associated to them. We study their relationship to Agler decompositions, regularity up to the boundary, and restriction maps into one variable spaces. We give a complete description of the important spaces associated to matrix rational inner functions. The dimension of some of these spaces can be computed in a straightforward way, and this ends up having an application to the study of three variable rational inner functions. Examples are included to highlight the differences between the scalar and matrix cases.

1. INTRODUCTION

Inner functions ϕ on the unit disk \mathbb{D} , their associated Hilbert spaces ϕH^2 and $\mathcal{H}_\phi = H^2 \ominus \phi H^2$, and the associated shift S and backward shift S^* operators on these spaces form a natural and rich area of analysis. Natural, because by Beurling's theorem [11], every invariant subspace of the forward shift on $\ell^2(\mathbb{N})$ is unitarily equivalent to ϕH^2 for some inner ϕ . Rich, because if we allow ϕ to be operator valued, then any contractive operator on a separable Hilbert space can be modeled as S^* on \mathcal{H}_ϕ for some ϕ . At the same time, simple choices for ϕ provide interesting examples. If ϕ is a finite Blaschke product, the space \mathcal{H}_ϕ is finite dimensional and is related to orthogonal polynomials on the unit circle. If $\phi(z) = \exp\left(a\frac{z+1}{z-1}\right)$, $a > 0$, then \mathcal{H}_ϕ is isometric to the Paley-Wiener space $PW_a = \mathcal{F}(L^2(0, a))$ through a change of coordinates to the upper half-plane. See [18] or [28] for the one variable theory.

Inner functions on the bidisk $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ and their associated Hilbert spaces are far richer and considerably less well developed than their one variable counterparts. For early work on the topic see for instance

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Rudin [29], Ahern-Clark [8], Ahern [7], and Sawyer [31]. Rational inner functions on the bidisk have close ties to the study of stable bivariate polynomials (e.g. polynomials with no zeros on the bidisk), and Hilbert space methods have proved useful in understanding them. See Cole-Wermer [13], Geronimo-Woerdeman [15], Ball-Sadosky-Vinnikov [10], Woerdeman [32], Kneese [22], and Geronimo-Iliev-Kneese [16]. Any type of general classification of inner functions on the bidisk or polydisk seems unknown and difficult.

Recall that $\phi : \mathbb{D}^2 \rightarrow \overline{\mathbb{D}}$ is an inner function if ϕ is holomorphic and satisfies

$$\lim_{r \nearrow 1} |\phi(re^{i\theta_1}, re^{i\theta_2})| = |\phi(e^{i\theta_1}, e^{i\theta_2})| = 1 \text{ a.e.}$$

We also use the term inner function for holomorphic functions $\phi : \mathbb{D}^2 \rightarrow \mathcal{B}_1$, where \mathcal{B}_1 is the closed unit ball in the operator norm of the bounded linear operators from a separable Hilbert space \mathcal{V} to itself such that

$$\phi(z)^* \phi(z) = \phi(z) \phi(z)^* = I \text{ for a.e. } z \in \mathbb{T}^2 = (\partial \mathbb{D})^2,$$

i.e. ϕ is unitary valued almost everywhere on the torus. Note that the radial boundary limits of these operator valued functions converge in the strong operator topology:

$$\lim_{r \nearrow 1} \phi(re^{i\theta_1}, re^{i\theta_2})v = \phi(e^{i\theta_1}, e^{i\theta_2})v$$

for each $v \in \mathcal{V}$ and for a.e. $(\theta_1, \theta_2) \in [0, 2\pi]^2$.

Let Z_1, Z_2 denote the coordinate functions $Z_j(z_1, z_2) = z_j$. Let us define some standard subspaces of $L^2 = L^2(\mathbb{T}^2) \otimes \mathcal{V}$ according to their Fourier series support. Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, $\mathbb{Z}_- = \{-1, -2, -3, \dots\}$. If $N \subset \mathbb{Z}^2$ and $f \in L^2$, the statement $\text{supp}(\hat{f}) \subset N$ means $\hat{f}(j_1, j_2) = 0$ for $(j_1, j_2) \notin N$. We caution that mention of \mathcal{V} is suppressed throughout our definitions in order to keep the notation uncluttered.

$$\begin{aligned} H^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \mathbb{Z}_+^2\} \\ L_{+\bullet}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \mathbb{Z}_+ \times \mathbb{Z}\} \\ L_{\bullet+}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \mathbb{Z} \times \mathbb{Z}_+\} \\ L_{-\bullet}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \mathbb{Z}_- \times \mathbb{Z}\} \\ L_{\bullet-}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \mathbb{Z} \times \mathbb{Z}_-\} \\ L_{+-}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \mathbb{Z}_+ \times \mathbb{Z}_-\} \\ L_{-+}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \mathbb{Z}_- \times \mathbb{Z}_+\} \\ L_{--}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \mathbb{Z}_- \times \mathbb{Z}_-\}. \end{aligned}$$

The important vector valued Hilbert spaces associated to ϕ are

$$\mathcal{H}_\phi = H^2 \ominus \phi H^2 = H^2 \cap \phi(L_{-+}^2 \oplus L_{+-}^2 \oplus L_{--}^2)$$

$$\mathcal{H}_\phi^1 = H^2 \cap \phi L_{\bullet-}^2$$

$$\mathcal{H}_\phi^2 = H^2 \cap \phi L_{-\bullet}^2$$

$$\mathcal{K}_\phi = H^2 \cap \phi L_{--}^2 = \mathcal{H}_\phi^1 \cap \mathcal{H}_\phi^2$$

$$\mathcal{K}_\phi^1 = H^2 \cap Z_1 \phi L_{--}^2$$

$$\mathcal{K}_\phi^2 = H^2 \cap Z_2 \phi L_{--}^2.$$

Example 1.1. The basic example $\phi(z) = z_1^2 z_2$ should help make these definitions more concrete. In this case, letting \vee denote closed linear span in H^2 , we have

$$\mathcal{H}_\phi = \vee\{Z_1^j Z_2^k : j, k \geq 0, \text{ and } j \leq 1 \text{ or } k = 0\}$$

$$\mathcal{H}_\phi^1 = \vee\{Z_1^j : j \geq 0\}$$

$$\mathcal{H}_\phi^2 = \vee\{Z_1^j Z_2^k : k \geq 0, j = 0, 1\}$$

$$\mathcal{K}_\phi = \vee\{1, Z_1\}$$

$$\mathcal{K}_\phi^1 = \vee\{1, Z_1, Z_1^2\}$$

$$\mathcal{K}_\phi^2 = \vee\{1, Z_1, Z_2, Z_1 Z_2\}.$$

◇

The space \mathcal{H}_ϕ seems to be the most natural generalization of the one variable space $H^2(\mathbb{T}) \ominus \phi H^2(\mathbb{T})$. For instance, in one variable, the reproducing kernel for $H^2 \ominus \phi H^2$ is given by

$$\frac{1 - \phi(z)\phi(w)^*}{1 - z\bar{w}},$$

while in two variables the reproducing kernel of \mathcal{H}_ϕ is

$$\frac{1 - \phi(z)\phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)}.$$

Unfortunately, this fact is not as illuminating as the one variable formula. The space \mathcal{H}_ϕ can be broken down into an orthogonal direct sum of various spaces above in a non-obvious way. This leads to a more useful formula for the reproducing kernel, a result of the work in Ball-Sadosky-Vinnikov [10] (see also [23]).

Notation 1.2. Let $\phi : \mathbb{D}^2 \rightarrow \mathcal{B}_1$ be inner. For $j = 1, 2$, let

$$E^j = \text{the reproducing kernel for } \mathcal{H}_\phi^j \ominus Z_j \mathcal{H}_\phi^j$$

$$F^j = \text{the reproducing kernel for } (\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi) \ominus Z_j(\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi)$$

$$G = \text{the reproducing kernel for } \mathcal{K}_\phi.$$

Theorem 1.3. *If $\phi : \mathbb{D}^2 \rightarrow \mathcal{B}_1$ is inner, then using Notation 1.2*

$$(1.1) \quad \frac{1 - \phi(z)\phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} = \frac{E^1(z, w)}{1 - z_1\bar{w}_1} + \frac{F^2(z, w)}{1 - z_2\bar{w}_2} \\ = \frac{F^1(z, w)}{1 - z_1\bar{w}_1} + \frac{F^2(z, w)}{1 - z_2\bar{w}_2} + G(z, w).$$

We shall outline a proof along the lines of Bickel [12], since various parts of the proof will be useful later. We will see later in Propositions 5.1 and 5.2 that the spaces in Notation 1.2 can be rewritten more simply

$$\mathcal{H}_\phi^j \ominus Z_j \mathcal{H}_\phi^j = \mathcal{K}_\phi^j \ominus Z_j \mathcal{K}_\phi \\ (\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi) \ominus Z_j (\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi) = \mathcal{K}_\phi^j \ominus \mathcal{K}_\phi.$$

Because of this, understanding ϕ boils down to understanding $\mathcal{K}_\phi, \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$, and $\mathcal{K}_\phi^2 \ominus \mathcal{K}_\phi$. *This is one of the most important themes of the paper.*

Rewriting the formula above we get

$$(1.2) \quad 1 - \phi(z)\phi(w)^* = (1 - z_1\bar{w}_1)F^2(z, w) + (1 - z_2\bar{w}_2)E^1(z, w),$$

which is called an *Agler decomposition* of ϕ . By symmetry we also have

$$(1.3) \quad 1 - \phi(z)\phi(w)^* = (1 - z_1\bar{w}_1)E^2(z, w) + (1 - z_2\bar{w}_2)F^1(z, w).$$

In the scalar case, one can deduce Agler's Pick interpolation theorem on the bidisk (see [1]) as well as Andô's inequality from operator theory from this formula via an approximation argument (see [22]).

Because of this connection, any pair of holomorphic positive semi-definite kernels (A^1, A^2) satisfying

$$(1.4) \quad 1 - \phi(z)\phi(w)^* = (1 - z_1\bar{w}_1)A^2(z, w) + (1 - z_2\bar{w}_2)A^1(z, w)$$

for all $z, w \in \mathbb{D}^2$ are called *Agler kernels* of ϕ . Labelling A^2 as the kernel next to $(1 - z_1\bar{w}_1)$ seems to be more natural in light of (1.1) and (1.2). The reproducing kernel Hilbert spaces associated to A^1, A^2 are denoted $\mathcal{H}(A^1)$ and $\mathcal{H}(A^2)$. Any such pair can be characterized in terms of the canonical kernels given in Theorem 1.3. This characterization generalizes a similar result of Ball-Sadosky-Vinnikov in [10].

To set things up, if we equate the right sides of (1.2), (1.3), and (1.4) one can derive

$$G^1(z, w) := \frac{A^1(z, w) - F^1(z, w)}{1 - z_1\bar{w}_1} = \frac{E^2(z, w) - A^2(z, w)}{1 - z_2\bar{w}_2} \\ G^2(z, w) := \frac{A^2(z, w) - F^2(z, w)}{1 - z_2\bar{w}_2} = \frac{E^1(z, w) - A^1(z, w)}{1 - z_1\bar{w}_1}.$$

Theorem 1.4. *Let $\phi : \mathbb{D}^2 \rightarrow \mathcal{B}_1$ be inner and let (A^1, A^2) be Agler kernels of ϕ . Then, G^1 and G^2 as above are both positive semidefinite and $G = G^1 + G^2$.*

Conversely, suppose G^1, G^2 are positive semidefinite kernels satisfying $G = G^1 + G^2$, and for $j = 1, 2$

$$A^j(z, w) := F^j(z, w) + (1 - z_j \bar{w}_j) G^j(z, w)$$

is positive semidefinite. Then, (A^1, A^2) are Agler kernels of ϕ .

The theorem says that in a certain sense E^j dominates any possible A^j , while F^j is dominated by any possible A^j .

One can also study the relationship between the boundary regularity of ϕ and the boundary regularity of functions in associated Hilbert spaces. In one variable, ϕ extends to be analytic at a boundary point if and only if every element of $H^2 \ominus \phi H^2$ does [18].

In [8], Ahern and Clark studied the relationship between regularity of an inner function ϕ on the boundary of the polydisk and regularity of elements of \mathcal{H}_ϕ . In particular, if every element of \mathcal{H}_ϕ extends holomorphically to a point $z \in \partial \mathbb{D}^n$ with $|z_k| = 1$ for some k , then ϕ depends on the k -th variable alone. This suggests \mathcal{H}_ϕ is too big to be of use in questions of regularity. The space \mathcal{K}_ϕ is not quite correct either, because it can be trivial even for rational inner functions.

For the remaining results, we restrict to finite-dimensional matrix valued inner functions. Define

$$\mathbb{E} = \mathbb{C} \setminus \overline{\mathbb{D}},$$

and then \mathbb{E}^2 will be what we call the exterior bidisk. If $z = (z_1, z_2) \in \mathbb{C}^2$, we sometimes write $1/\bar{z} = (1/\bar{z}_1, 1/\bar{z}_2)$ for short.

Theorem 1.5. *Let $\phi : \mathbb{D}^2 \rightarrow \mathcal{B}_1$ be a matrix valued inner function (i.e. finite dimensional matrix valued). Let X be an open subset of \mathbb{T}^2 and let*

$$X_1 = \{x_1 \in \mathbb{T} : \exists x_2 \in \mathbb{T} \text{ with } (x_1, x_2) \in X\}$$

$$X_2 = \{x_2 \in \mathbb{T} : \exists x_1 \in \mathbb{T} \text{ with } (x_1, x_2) \in X\}$$

$$S = \{1/\bar{z} : \det \phi(z) = 0\}.$$

Then the following are equivalent:

- (i) *The function ϕ extends continuously to X .*
- (ii) *For some pair (A^1, A^2) of Agler kernels of ϕ , the elements of $\mathcal{H}(A^1), \mathcal{H}(A^2)$ extend continuously to X .*
- (iii) *There is a domain Ω containing*

$$\mathbb{D}^2 \cup X \cup (X_1 \times \mathbb{D}) \cup (\mathbb{D} \times X_2) \cup (\mathbb{E}^2 \setminus S)$$

on which ϕ and the elements of $\mathcal{K}_\phi, \mathcal{K}_\phi^1, \mathcal{K}_\phi^2$ extend to be analytic (and meromorphic on $\Omega \cup S$). Point evaluation in Ω is bounded in these spaces, and therefore, the kernels G, F^1, F^2, E^1, E^2 and all Agler kernels (A^1, A^2) of ϕ extend to be sesqui-analytic on $\Omega \times \Omega$.

Continuity up to the boundary is a strong requirement. For weaker notions of extension to the boundary and their relations to Agler decompositions, see [4] and [5] by Agler, McCarthy, and Young.

Restriction maps of canonical spaces are not only bounded, they are also isometries into one variable spaces.

Theorem 1.6. *Let $\phi : \mathbb{D}^2 \rightarrow \mathcal{B}_1$ be a matrix valued inner function. For almost every $t \in \mathbb{T}$, the map*

$$f \mapsto f(t, \cdot)$$

embeds $\mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi$ and $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ isometrically into $H^2(\mathbb{T}) \ominus \phi(t, \cdot) H^2(\mathbb{T})$.

Finally, we can give an explicit description of the spaces involved for a rational inner function $\phi = Q/p$, where Q is an $N \times N$ matrix polynomial and $p \in \mathbb{C}[z_1, z_2]$ has no zeros in \mathbb{D}^2 . If Q has degree $d = (d_1, d_2)$, define

$$\begin{aligned} \mathcal{P}_{d,p}^0 &= \{q/p \in H^2 : \deg q \leq (d_1 - 1, d_2 - 1)\} \\ \mathcal{P}_{d,p}^1 &= \{q/p \in H^2 : \deg q \leq (d_1, d_2 - 1)\} \\ \mathcal{P}_{d,p}^2 &= \{q/p \in H^2 : \deg q \leq (d_1 - 1, d_2)\}. \end{aligned}$$

The requirement that $q/p \in L^2$ is analytic in the sense that the zeros of q must counter those of p . Also, define $\tilde{Q}(z) = z^d Q(1/\bar{z})^*$. It turns out $\tilde{\phi} = \tilde{Q}/p$ is a rational inner function, and we can give the following analytic/algebraic description of the spaces associated to ϕ .

Theorem 1.7. *If $\phi = Q/p$ is a rational matrix inner function, then*

$$\begin{aligned} \mathcal{K}_\phi &= \{f \in \mathcal{P}_{d,p}^0 : \tilde{\phi}f \in \mathcal{P}_{d,p}^0\} \\ \mathcal{K}_\phi^1 &= \{f \in \mathcal{P}_{d,p}^1 : \tilde{\phi}f \in \mathcal{P}_{d,p}^1\} \\ \mathcal{K}_\phi^2 &= \{f \in \mathcal{P}_{d,p}^2 : \tilde{\phi}f \in \mathcal{P}_{d,p}^2\}. \end{aligned}$$

Notice in particular that these spaces are all finite dimensional, something shown already in Ball-Sadosky-Vinnikov [10] when ϕ is regular up to \mathbb{T}^2 . The significance here is that we have given a complete description of the spaces involved even when there are singularities on \mathbb{T}^2 (already done in the scalar case in [22]), and in addition, some of the dimensions involved can be determined in a straightforward way.

Write $\det \phi = \frac{\tilde{g}}{g}$, where g is a polynomial with no zeros on \mathbb{D}^2 and no factors in common with \tilde{g} .

Theorem 1.8. *With the same setup as the previous theorem,*

$$\begin{aligned} \dim \mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi &= \dim \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi = \deg_2 \tilde{g} \\ \dim \mathcal{K}_\phi^2 \ominus Z_2 \mathcal{K}_\phi &= \dim \mathcal{K}_\phi^2 \ominus \mathcal{K}_\phi = \deg_1 \tilde{g}. \end{aligned}$$

The dimension of \mathcal{K}_ϕ depends on the nature of the zeros of p on \mathbb{T}^2 and properties of Q and therefore cannot be given by such a simple formula. See Example 10.1.

It is then possible to show that the canonical Agler kernels (E^1, F^2) or (F^1, E^2) have minimal dimensions.

Corollary 1.9. *Let ϕ be an $N \times N$ matrix valued rational inner function on \mathbb{D}^2 . Write $\det \phi = \tilde{g}/g$ in lowest terms and let $d = (d_1, d_2) = \deg \tilde{g}$. If (A^1, A^2) are Agler kernels, then*

$$\dim \mathcal{H}(A^1) \geq d_2 \text{ and } \dim \mathcal{H}(A^2) \geq d_1.$$

Remark 1.10. We point out some known applications of the above work in the scalar case of $\phi = \tilde{p}/p$.

The existence of Agler kernels with minimal dimensions was important in proving one of the main results in Agler-McCarthy-Young [6].

The details of the canonical spaces and kernels have been important in understanding function theory on *distinguished varieties* in [21], [20]. Distinguished varieties are algebraic curves in \mathbb{C}^2 which exit the bidisk through the distinguished boundary [2].

Since the spaces $\mathcal{K}_\phi, \mathcal{K}_\phi^1, \mathcal{K}_\phi^2$ consist of rational functions with denominator p , orthogonality relations in these spaces can be reinterpreted as orthogonality relations between spaces of polynomials using the inner product of $L^2(\frac{1}{|p|^2} d\sigma, \mathbb{T}^2)$. In particular, the fact that

$$\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi \perp \mathcal{K}_\phi^2$$

can be reinterpreted as the key condition required to characterize measures of the form $\frac{1}{|p|^2} d\sigma$ on \mathbb{T}^2 in Geronimo-Woerdeman [15]. This condition can further be used to give a characterization of positive two variable trigonometric polynomials t which can be written as $|p(z, w)|^2$, where p is a polynomial with no zeros on $\overline{\mathbb{D}^2}$.

The following result of Kummert [26] on transfer function representations is a direct consequence of Theorem 1.8 (see also Ball-Sadosky-Vinnikov [10]).

Corollary 1.11. *Let ϕ be an $N \times N$ matrix valued rational inner function on \mathbb{D}^2 . Write $\det \phi = \tilde{g}/g$ in lowest terms and let $d = (d_1, d_2) =$*

$\deg \tilde{g}$, $|d| = d_1 + d_2$. Then, there exists an $(N + |d|) \times (N + |d|)$ unitary matrix U , which we write in block form

$$U = \begin{array}{cc} \mathbb{C}^N & \mathbb{C}^{|d|} \\ \mathbb{C}^{|d|} & \end{array} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{array}{cc} \mathbb{C}^N & \\ \mathbb{C}^{d_1} & \mathbb{C}^{d_2} \end{array} \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix}$$

such that

$$(1.5) \quad \phi(z) = A + Bd(z)(I - Dd(z))^{-1}C,$$

$$\text{where } d(z) = \begin{pmatrix} z_1 I_{d_1} & 0 \\ 0 & z_2 I_{d_2} \end{pmatrix}.$$

Furthermore, $(N + |d|) \times (N + |d|)$ is the minimum possible size of such a representation.

It is a standard calculation that given a unitary U , (1.5) yields a matrix rational inner function, so the transfer function representation gives a way to write down every rational inner function, although the representation may not be unique.

We have emphasized Agler decompositions with minimal dimensions because of the following application of Theorem 1.8 to the study of *three* variable rational inner functions. We offer an improvement to a result of Kneese [25], which in turn was a generalization of a result of Kummert [27].

Theorem 1.12. *Let $p \in \mathbb{C}[z_1, z_2, z_3]$ have degree $(n, 1, 1)$ and no zeros on $\overline{\mathbb{D}^3}$, let $\tilde{p}(z) = z_1^n z_2 z_3 \overline{p(1/\bar{z})}$, and define the rational inner function $\phi = \tilde{p}/p$. Then, ϕ has an Agler decomposition of the form*

$$1 - \phi(z)\overline{\phi(w)} = \sum_{j=1}^3 (1 - z_j \bar{w}_j) SOS_j(z, w),$$

where SOS_2 and SOS_3 are sums of two squares, while SOS_1 is a sum of $2n$ squares.

To be clear, $SOS_2(z, z) = \frac{1}{|p(z)|^2} (|A_1(z)|^2 + |A_2(z)|^2)$ for some polynomials $A_1, A_2 \in \mathbb{C}[z_1, z_2, z_3]$, and similarly for SOS_3 , while $SOS_1(z, z) = \frac{1}{|p(z)|^2} \sum_{j=1}^{2n} |B_j(z)|^2$ for some polynomials $B_1, \dots, B_{2n} \in \mathbb{C}[z_1, z_2, z_3]$. Part of the significance of the result is that no such decomposition can generally exist for rational inner functions on \mathbb{D}^3 (regardless of the bounds on the number of squares involved). For more on this, see [25].

On the other hand, part of the significance of the result is the bounds obtained on the number of squares involved. The original theorem in

[25] had the non-optimal bounds of $4n^1$, $2(n+1)$, 2 for the number of squares in SOS_1, SOS_2, SOS_3 .

In [24], an explicit Agler decomposition was found for \tilde{p}/p when $p(z_1, z_2, z_3) = 3 - z_1 - z_2 - z_3$ with sums of squares terms SOS_1, SOS_2, SOS_3 containing 3 squares each. Although p has a zero on \mathbb{T}^3 , the proof for Theorem 1.12 works for p (and in fact should work for any p with no zeros on \mathbb{D}^3 when p and \tilde{p} have no factors in common). We conclude that \tilde{p}/p has an Agler decomposition with 2 squares in each sums of squares term. It is shown in [24] that none of the terms SOS_1, SOS_2, SOS_3 can be written as a single square. This shows Theorem 1.12 is optimal when $n = 1$.

Question 1.13. *Is Theorem 1.12 optimal for $n > 1$? Namely, is there a rational inner function of degree $(n, 1, 1)$ ($n > 1$) such that for every Agler decomposition, SOS_1 is a sum of $2n$ or more squares?*

The rest of the paper is summarized in the table of contents.

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2. THEOREM 1.3 ON FUNDAMENTAL AGLER DECOMPOSITIONS

In this section, we sketch the proof of Theorem 1.3.

¹Regrettably, due to an arithmetic error $4(n-1)$ was written in the original paper [25] instead of $4n$.

We first note some simple inclusions

$$\mathcal{K}_\phi, Z_j \mathcal{K}_\phi \subset \mathcal{K}_\phi^j \subset \mathcal{H}_\phi^j \subset \mathcal{H}_\phi.$$

The spaces $\mathcal{K}_\phi, \mathcal{K}_\phi^j$ should be thought of as “small” since they are finite dimensional in the case of rational ϕ .

The following can be proved straight from definitions.

Proposition 2.1. *For $j = 1, 2$, the space \mathcal{H}_ϕ^j is invariant under multiplication by Z_j , and even more, if $f \in \mathcal{H}_\phi$ and if $Z_j^k f \in \mathcal{H}_\phi$ for all $k > 0$, then $f \in \mathcal{H}_\phi^j$.*

Hence, \mathcal{H}_ϕ^j is maximal among all Z_j invariant subspaces of \mathcal{H}_ϕ . As ϕ is unitary valued a.e. on \mathbb{T}^2 , it follows that $\phi L_{\bullet-}^2 = (\phi L_{\bullet+}^2)^\perp$. An observation in [12] is that

$$\mathcal{H}_\phi^1 = H^2 \cap (\phi L_{\bullet+}^2)^\perp = H^2 \cap (\phi H^2)^\perp \cap (\phi L_{-+}^2)^\perp = \mathcal{H}_\phi \cap (\phi L_{-+}^2)^\perp,$$

so that we have the following:

Proposition 2.2.

$$\mathcal{H}_\phi^1 = \mathcal{H}_\phi \ominus P_{\mathcal{H}_\phi}(\phi L_{-+}^2)$$

and so

$$\mathcal{H}_\phi \ominus \mathcal{H}_\phi^1 = \overline{P_{\mathcal{H}_\phi}(\phi L_{-+}^2)} = \overline{P_{H^2}(\phi L_{-+}^2)} = \overline{P_{L_{\bullet+}^2}(\phi L_{-+}^2)}.$$

Here P denotes orthogonal projection onto the space in the subscript and the overset bars denote closures. The second equality follows from the fact that $\phi H^2 \perp \phi L_{-+}^2$. The last equality follows from the fact that $\phi L_{-+}^2 \subset L_{\bullet+}^2$ and therefore projecting onto either H^2 or $L_{\bullet+}^2$ has the same effect.

It is easy to show that $P_{L_{\bullet+}^2}(\phi L_{-+}^2)$ is invariant under multiplication by Z_2 since ϕ is in H^∞ . So, $\mathcal{H}_\phi \ominus \mathcal{H}_\phi^1$ is invariant under multiplication by Z_2 , and therefore $\mathcal{H}_\phi \ominus \mathcal{H}_\phi^1 \subset \mathcal{H}_\phi^2$ by Proposition 2.1. By Lemma 2.6 below and the fact $\mathcal{H}_\phi^1 \cap \mathcal{H}_\phi^2 = \mathcal{K}_\phi$, we immediately have the following two propositions.

Proposition 2.3.

$$\mathcal{H}_\phi \ominus \mathcal{H}_\phi^1 = \mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi$$

so that

$$\mathcal{H}_\phi = (\mathcal{H}_\phi^1 \ominus \mathcal{K}_\phi) \oplus (\mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi) \oplus \mathcal{K}_\phi.$$

Proposition 2.4. *For $j = 1, 2$, $\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi$ is invariant under multiplication by Z_j .*

The following is a standard fact.

Proposition 2.5. *The space \mathcal{H}_ϕ is a reproducing kernel Hilbert space with reproducing kernel*

$$\frac{1 - \phi(z)\phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)}$$

for $z, w \in \mathbb{D}^2$.

Sketch of Proof of Theorem 1.3. Multiplication by Z_j is a pure isometry on H^2 and hence on \mathcal{H}_ϕ^j and $\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi$. By Propositions 2.1 and 2.4, we have the orthogonal decompositions

$$\mathcal{H}_\phi^j = \bigoplus_{k \geq 0} Z_j^k (\mathcal{H}_\phi^j \ominus Z_j \mathcal{H}_\phi^j) \text{ and } \mathcal{H}_\phi^j \ominus \mathcal{K}_\phi = \bigoplus_{k \geq 0} Z_j^k ((\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi) \ominus Z_j (\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi)).$$

Because of this, the reproducing kernels for \mathcal{H}_ϕ^j and $\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi$ are given by

$$\frac{E^j(z, w)}{1 - z_j\bar{w}_j} \text{ and } \frac{F^j(z, w)}{1 - z_j\bar{w}_j}$$

respectively (recall Notation 1.2). By Proposition 2.3,

$$\mathcal{H}_\phi = \mathcal{H}_\phi^1 \oplus (\mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi) = (\mathcal{H}_\phi^1 \ominus \mathcal{K}_\phi) \oplus (\mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi) \oplus \mathcal{K}_\phi.$$

Therefore, the reproducing kernel for \mathcal{H}_ϕ can be decomposed in the following two ways:

$$\begin{aligned} \frac{1 - \phi(z)\phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} &= \frac{E^1(z, w)}{1 - z_1\bar{w}_1} + \frac{F^2(z, w)}{1 - z_2\bar{w}_2} \\ &= \frac{F^1(z, w)}{1 - z_1\bar{w}_1} + \frac{F^2(z, w)}{1 - z_2\bar{w}_2} + G(z, w). \end{aligned}$$

□

The following general Hilbert space lemma was used above. It makes a few arguments later easier to digest.

Lemma 2.6. *Let \mathcal{H} be a Hilbert space with (closed) subspaces $\mathcal{K}_1, \mathcal{K}_2$. If $\mathcal{H} \ominus \mathcal{K}_1 \subset \mathcal{K}_2$, then*

$$\mathcal{H} \ominus \mathcal{K}_1 = \mathcal{K}_2 \ominus (\mathcal{K}_1 \cap \mathcal{K}_2).$$

Proof. The inclusion \subset is trivial. For the opposite direction, suppose $f \in \mathcal{K}_2 \ominus (\mathcal{K}_1 \cap \mathcal{K}_2)$ and $f \perp \mathcal{H} \ominus \mathcal{K}_1$. Then, $f \in \mathcal{K}_1$ and $f \in \mathcal{K}_2$ making f orthogonal to itself. Hence, $f = 0$. □

3. TWO GENERAL LEMMAS ON REPRODUCING KERNELS

We record two standard facts about reproducing kernel Hilbert spaces. See [9] or [1] for more general information.

Let $B(\mathcal{V})$ be the bounded linear operators on \mathcal{V} , our separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$. Given a function $H : X \times X \rightarrow B(\mathcal{V})$, the notation

$$H \succcurlyeq 0$$

means H is a positive semidefinite kernel. Namely, for any $x_1, \dots, x_N \in X$, the block operator on \mathcal{V}^N

$$(H(x_j, x_k))_{jk}$$

is positive semidefinite. In addition, for $K : X \times X \rightarrow B(\mathcal{V})$, the notation $H \succcurlyeq K$ means $H - K \succcurlyeq 0$. We write $\mathcal{H}(K)$ for the canonical reproducing kernel Hilbert space of \mathcal{V} valued functions on X associated to a positive semidefinite kernel K , and $\langle \cdot, \cdot \rangle_{\mathcal{H}(K)}$ denotes the inner product in $\mathcal{H}(K)$. The reproducing kernels are $K_x(\cdot)v = K(\cdot, x)v$ for $x \in X, v \in \mathcal{V}$, by which we mean

$$\langle f, K_x v \rangle_{\mathcal{H}(K)} = \langle f(x), v \rangle_{\mathcal{V}}$$

for any $f \in \mathcal{H}(K)$. The span of these functions is dense in $\mathcal{H}(K)$.

Remark 3.1. The setting of vector valued functions and operator valued kernels can easily be reduced to the setting found in standard references of scalar functions and kernels by viewing our “points” as elements of $X \times \mathcal{V}$ as opposed to X . Evaluating $f \in \mathcal{H}(K)$ at the “point” (x, v) would refer to $\langle f(x), v \rangle_{\mathcal{V}}$.

It is a standard fact that a function $f : X \rightarrow \mathcal{V}$ is in $\mathcal{H}(K)$ if and only if there is an $\alpha \geq 0$ such that

$$\alpha K(y, x) \succcurlyeq f(y)f(x)^*,$$

where for any $v \in \mathcal{V}$, v^* denotes the functional $\langle \cdot, v \rangle_{\mathcal{V}}$. The minimum of all such α 's is $\|f\|_{\mathcal{H}(K)}^2$. So, if $H \succcurlyeq K \succcurlyeq 0$ and if $f \neq 0$, then

$$H(y, x) \succcurlyeq K(y, x) \succcurlyeq \frac{f(y)f(x)^*}{\|f\|_{\mathcal{H}(K)}^2}.$$

This implies $\|f\|_{\mathcal{H}(H)} \leq \|f\|_{\mathcal{H}(K)}$, which gives the following lemma:

Lemma 3.2. *If $H \succcurlyeq K \succcurlyeq 0$ on X , then $\mathcal{H}(K) \subseteq \mathcal{H}(H)$, and the embedding $\iota : \mathcal{H}(K) \rightarrow \mathcal{H}(H)$ is a contraction.*

If $F \subset X$ is a finite set and $v : F \rightarrow \mathcal{V}$ is a function, then

$$\left\| \sum_{x \in F} K_x v(x) \right\|_{\mathcal{H}(K)}^2 = \sum_{x, y \in F} \langle K(y, x)v(x), v(y) \rangle_{\mathcal{V}}$$

essentially by definition of the inner product in $\mathcal{H}(K)$. Set $f = \sum_{x \in F} K_x v(x)$. So, if $H \succcurlyeq K \succcurlyeq 0$, then $f \in \mathcal{H}(H)$ and $\|f\|_{\mathcal{H}(H)}^2 \leq \|f\|_{\mathcal{H}(K)}^2$, i.e.

$$\sum_{x,y \in F} \langle K_x v(x), K_y v(y) \rangle_{\mathcal{H}(H)} \leq \sum_{x,y \in F} \langle K(y, x) v(x), v(y) \rangle_{\mathcal{V}}.$$

We also need the following:

Lemma 3.3. *Let H be a positive semidefinite kernel on X , and let \mathcal{K} be a closed subspace of $\mathcal{H}(H)$ with reproducing kernel K . Suppose $H \succcurlyeq L \succcurlyeq 0$ and $L_x(\cdot)v = L(\cdot, x)v \in \mathcal{K}$ for all $x \in X, v \in \mathcal{V}$. Then, $K \succcurlyeq L$.*

Proof. Let $F \subset X$ be a finite set and $v : F \rightarrow \mathcal{V}$ a function. Define $f = \sum_{x \in F} L_x v(x)$, $g = \sum_{x \in F} K_x v(x)$. Then, $f, g \in \mathcal{K}$ and $f \in \mathcal{H}(L)$. Therefore,

$$\begin{aligned} \|f\|_{\mathcal{H}(L)}^2 &= \sum_{x,y \in F} \langle L(y, x) v(x), v(y) \rangle_{\mathcal{V}} \\ &= \left\langle \sum_{x \in F} L_x v(x), \sum_{y \in F} K_y v(y) \right\rangle_{\mathcal{H}(H)} \\ &= \langle f, g \rangle_{\mathcal{H}(H)} \\ &\leq \|f\|_{\mathcal{H}(H)} \|g\|_{\mathcal{H}(H)} \\ &\leq \|f\|_{\mathcal{H}(L)} \|g\|_{\mathcal{H}(H)} \text{ by Lemma 3.2,} \end{aligned}$$

which shows $\|f\|_{\mathcal{H}(L)}^2 \leq \|g\|_{\mathcal{H}(H)}^2$. Expanding this out shows

$$\sum_{x,y \in F} \langle L(y, x) v(x), v(y) \rangle_{\mathcal{V}} \leq \sum_{x,y \in F} \langle K(y, x) v(x), v(y) \rangle_{\mathcal{V}},$$

which shows $L \preccurlyeq K$. □

4. THEOREM 1.4 ON MAXIMALITY AND MINIMALITY

The “canonical” Agler decompositions from Theorem 1.3 are maximal and minimal in the sense described in the following theorem. Moreover, all other Agler decompositions can be characterized in terms of properties of F^1, F^2, G .

This maximality and minimality property is found in Theorem 5.5 of [10]. The following result is more general in one sense; we consider Agler decompositions which do not necessarily come from an orthogonal decomposition inside H^2 .

This generality is nontrivial. Specifically, by considering monomial inner functions like $\phi(z) = z_1^2 z_2$, one can show there are Agler decompositions that cannot be written as convex combinations of Agler

decompositions coming from orthogonal decompositions inside H^2 . See Example 10.4.

Theorem 1.4. *Let $\phi : \mathbb{D}^2 \rightarrow \mathcal{B}_1$ be inner and let (A^1, A^2) be Agler kernels of ϕ . Then, for $j, k \in \{1, 2\}$ distinct*

$$G^j(z, w) := \frac{A^j(z, w) - F^j(z, w)}{1 - z_j \bar{w}_j} = \frac{E^k(z, w) - A^k(z, w)}{1 - z_k \bar{w}_k}$$

is positive semidefinite and

$$G(z, w) = G^1(z, w) + G^2(z, w).$$

Conversely, suppose G^1, G^2 are positive semidefinite and satisfy $G = G^1 + G^2$, while for $j = 1, 2$,

$$A^j(z, w) := F^j(z, w) + (1 - z_j \bar{w}_j)G^j(z, w)$$

is positive semidefinite. Then, (A^1, A^2) are Agler kernels of ϕ .

For a quick corollary, observe that if $G = 0$, or equivalently, $\mathcal{K}_\phi = \{0\}$, then ϕ has a unique Agler decomposition. On the other hand, if ϕ has a unique Agler decomposition, then $E^2 = F^2$, $E^1 = F^1$, and then $G = 0$, yielding the following:

Corollary 4.1. *An inner function ϕ has a unique Agler decomposition if and only if $\mathcal{K}_\phi = \{0\}$.*

Proof of Theorem. Set $L(z, w) = A^1(z, w)/(1 - z_1 \bar{w}_1)$. Since

$$\frac{1 - \phi(z)\phi(w)^*}{(1 - z_1 \bar{w}_1)(1 - z_2 \bar{w}_2)} = \frac{A^1(z, w)}{1 - z_1 \bar{w}_1} + \frac{A^2(z, w)}{1 - z_2 \bar{w}_2} \succcurlyeq L(z, w),$$

Lemma 3.2 implies that $\mathcal{H}(L) \subseteq \mathcal{H}_\phi$. In addition,

$$(1 - z_1 \bar{w}_1)L(z, w) = A^1(z, w) \succcurlyeq 0$$

shows $\mathcal{H}(L)$ is invariant under multiplication by Z_1 . In particular, $Z_1^j L_w v \in \mathcal{H}(L) \subseteq \mathcal{H}_\phi$ for all $j \geq 0$, $w \in \mathbb{D}^2$, $v \in \mathcal{V}$. Then, by Proposition 2.1, each $L_w v \in \mathcal{H}_\phi^1$. It follows from Lemma 3.3 that

$$\frac{E^1(z, w)}{1 - z_1 \bar{w}_1} \succcurlyeq \frac{A^1(z, w)}{1 - z_1 \bar{w}_1},$$

and so $G^2(z, w) \succcurlyeq 0$. The remainder of the forward implication follows from algebraic manipulations.

For the converse, we immediately have

$$\begin{aligned}
& (1 - z_1 \bar{w}_1)A^2(z, w) + (1 - z_2 \bar{w}_2)A^1(z, w) \\
&= (1 - z_1 \bar{w}_1)(F^2(z, w) + (1 - z_2 \bar{w}_2)G^2(z, w)) \\
&\quad + (1 - z_2 \bar{w}_2)(F^1(z, w) + (1 - z_1 \bar{w}_1)G^1(z, w)) \\
&= (1 - z_1 \bar{w}_1)F^2(z, w) + (1 - z_2 \bar{w}_2)F^1(z, w) \\
&\quad + (1 - z_1 \bar{w}_1)(1 - z_2 \bar{w}_2)G(z, w) \\
&= 1 - \phi(z)\phi(w)^*,
\end{aligned}$$

so that (A^1, A^2) are Agler kernels of ϕ . \square

5. MORE DETAILS ON THE CANONICAL SUBSPACES

The previous sections show that the study of an inner function in two variables hinges on the subspaces:

$$(\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi) \ominus Z_j(\mathcal{H}_\phi^j \ominus \mathcal{K}_\phi) \text{ for } j = 1, 2$$

$$\mathcal{H}_\phi^j \ominus Z_j \mathcal{H}_\phi^j \text{ for } j = 1, 2$$

and \mathcal{K}_ϕ . The main point of this section is that all of these subspaces are “small” in the sense that they sit inside either \mathcal{K}_ϕ^1 or \mathcal{K}_ϕ^2 . Let

$$\begin{aligned}
L_{0-}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \{0\} \times \mathbb{Z}_-\} \\
L_{0+}^2 &= \{f \in L^2 : \text{supp}(\hat{f}) \subset \{0\} \times \mathbb{Z}_+\}
\end{aligned}$$

and define L_{-0}^2 and L_{+0}^2 similarly. We shall use $A \vee B$ to denote the closed linear span of two sets A and B in a common Hilbert space.

Proposition 5.1.

$$(\mathcal{H}_\phi^1 \ominus \mathcal{K}_\phi) \ominus Z_1(\mathcal{H}_\phi^1 \ominus \mathcal{K}_\phi) = \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi = \overline{P_{\mathcal{K}_\phi^1}(\phi L_{0-}^2)}$$

$$(\mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi) \ominus Z_2(\mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi) = \mathcal{K}_\phi^2 \ominus \mathcal{K}_\phi = \overline{P_{\mathcal{K}_\phi^2}(\phi L_{-0}^2)}.$$

Proof. We define the two subspaces \mathcal{Q} and \mathcal{R} :

$$\begin{aligned}
\mathcal{R} &= L_{+\bullet}^2 \cap \phi L_{--}^2 \\
\mathcal{Q} &= L_{+\bullet}^2 \cap \phi L_{-\bullet}^2 = L_{+\bullet}^2 \ominus \phi L_{+\bullet}^2
\end{aligned}$$

and calculate

$$\begin{aligned}
\mathcal{Q} \ominus \mathcal{R} &= P_{\mathcal{Q}}(\mathcal{R}^\perp) \\
&= P_{\mathcal{Q}}(L_{-\bullet}^2 \vee \phi(L_{-+}^2 \oplus L_{+\bullet}^2)) \\
&= \overline{P_{\mathcal{Q}}(L_{-\bullet}^2 + \phi(L_{-+}^2 \oplus L_{+\bullet}^2))} \\
&= \overline{P_{\mathcal{Q}}(\phi L_{-+}^2)} && \text{since } \mathcal{Q} \perp L_{-\bullet}^2, \phi L_{+\bullet}^2 \\
&= \overline{(P_{\mathcal{Q}} + P_{\phi L_{+\bullet}^2})(\phi L_{-+}^2)} && \text{since } P_{\phi L_{+\bullet}^2}(\phi L_{-+}^2) = 0 \\
&= \overline{P_{L_{+\bullet}^2}(\phi L_{-+}^2)} && \text{since } L_{+\bullet}^2 = \mathcal{Q} \oplus \phi L_{+\bullet}^2 \\
&= \mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi && \text{by Proposition 2.2.}
\end{aligned}$$

Therefore, the “wandering” subspace satisfies

$$\begin{aligned}
(\mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi) \ominus Z_2(\mathcal{H}_\phi^2 \ominus \mathcal{K}_\phi) &= (\mathcal{Q} \ominus \mathcal{R}) \ominus Z_2(\mathcal{Q} \ominus \mathcal{R}) \\
&= (\mathcal{Q} \ominus \mathcal{R}) \ominus (\mathcal{Q} \ominus Z_2\mathcal{R}) && \text{since } Z_2\mathcal{Q} = \mathcal{Q} \\
&= Z_2\mathcal{R} \ominus \mathcal{R} \subset \mathcal{H}_\phi^2.
\end{aligned}$$

Recall that $Z_2\mathcal{R} \cap \mathcal{H}_\phi^2 = H^2 \cap Z_2\phi L_{--}^2 = \mathcal{K}_\phi^2$. Using Lemma 2.6, we now intersect with \mathcal{H}_ϕ^2 to obtain

$$\begin{aligned}
Z_2\mathcal{R} \ominus \mathcal{R} &= (Z_2\mathcal{R} \cap \mathcal{H}_\phi^2) \ominus (\mathcal{R} \cap \mathcal{H}_\phi^2) \\
&= \mathcal{K}_\phi^2 \ominus \mathcal{K}_\phi
\end{aligned}$$

equals the wandering subspace. We can also identify $\mathcal{K}_\phi^2 \ominus \mathcal{K}_\phi$ with a “closure of a projection” as follows:

$$\begin{aligned}
\mathcal{K}_\phi^2 \ominus \mathcal{K}_\phi &= P_{\mathcal{K}_\phi^2}((\mathcal{K}_\phi)^\perp) \\
&= P_{\mathcal{K}_\phi^2}((L^2 \ominus H^2) \vee \phi(L_{+\bullet}^2 \oplus L_{-+}^2)) \\
&= \overline{P_{\mathcal{K}_\phi^2}((L^2 \ominus H^2) + \phi(L_{+\bullet}^2 \oplus L_{-+}^2))} \\
&= \overline{P_{\mathcal{K}_\phi^2}(\phi L_{-0}^2)},
\end{aligned}$$

since $\mathcal{K}_\phi^2 \subset Z_2\phi L_{--}^2 \perp \phi(L_{+\bullet}^2 \oplus Z_2L_{-+}^2)$. □

Proposition 5.2.

$$\begin{aligned}
\mathcal{H}_\phi^1 \ominus Z_1\mathcal{H}_\phi^1 &= \mathcal{K}_\phi^1 \ominus Z_1\mathcal{K}_\phi = \overline{P_{\mathcal{K}_\phi^1}(L_{0+}^2)} \\
\mathcal{H}_\phi^2 \ominus Z_2\mathcal{H}_\phi^2 &= \mathcal{K}_\phi^2 \ominus Z_2\mathcal{K}_\phi = \overline{P_{\mathcal{K}_\phi^2}(L_{+0}^2)}.
\end{aligned}$$

Proof. Since

$$E^1(z, w) = F^1(z, w) + (1 - z_1\bar{w}_1)G(z, w)$$

and since $F_w^1 v \in \mathcal{K}_\phi^1$ for $w \in \mathbb{D}^2, v \in \mathcal{V}$, we see that $E_w^1 v \in \mathcal{K}_\phi^1$. Hence, $\mathcal{H}_\phi^1 \ominus Z_1 \mathcal{H}_\phi^1 \subset \mathcal{K}_\phi^1$, so by Lemma 2.6

$$\mathcal{H}_\phi^1 \ominus Z_1 \mathcal{H}_\phi^1 = \mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi.$$

We can also identify $\mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi$ with a “closure of a projection” because

$$\begin{aligned} \mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi &= P_{\mathcal{K}_\phi^1}((Z_1 \mathcal{K}_\phi)^\perp) \\ &= P_{\mathcal{K}_\phi^1}((L^2 \ominus Z_1 H^2) \vee Z_1 \phi(L_{+\bullet}^2 \oplus L_{-+}^2)) \\ &= \overline{P_{\mathcal{K}_\phi^1}((L^2 \ominus Z_1 H^2) + Z_1 \phi(L_{+\bullet}^2 \oplus L_{-+}^2))} \\ &= \overline{P_{\mathcal{K}_\phi^1}(L_{0+}^2)}. \end{aligned}$$

□

The characterization in Theorem 1.4 implies that the reproducing kernel Hilbert spaces associated to any Agler decomposition must also sit inside either \mathcal{K}_ϕ^1 or \mathcal{K}_ϕ^2 .

Corollary 5.3. *Let (A^1, A^2) be Agler kernels of ϕ . Then*

$$\mathcal{H}(A^j) \text{ is contained contractively in } \mathcal{K}_\phi^j \text{ for } j = 1, 2.$$

Proof. By Theorem 1.4, for $j = 1, 2$, we can write

$$A^j(z, w) = F^j(z, w) + (1 - z_j \bar{w}_j) G^j(z, w),$$

where each G^j is positive semidefinite and $G = G^1 + G^2$. From Proposition 5.1 and the definitions of F^j and G , it is clear that \mathcal{K}_ϕ^j has reproducing kernel $F^j + G$. As

$$F^1 + G - A^1 = G^2 + z_1 \bar{w}_1 G^1 \succcurlyeq 0,$$

it follows from Lemma 3.2 that $\mathcal{H}(A^1)$ is contained contractively in $\mathcal{H}(F^1 + G) = \mathcal{K}_\phi^1$ and similarly, $\mathcal{H}(A^2)$ is in \mathcal{K}_ϕ^2 . □

6. MATRIX INNER FUNCTIONS AND THEOREM 1.5 ON REGULARITY

For the rest of the paper we assume $\mathcal{V} = \mathbb{C}^N$ and therefore, that ϕ is an $N \times N$ matrix valued inner function on \mathbb{D}^2 . Define

$$\mathbb{E} = \mathbb{C} \setminus \overline{\mathbb{D}},$$

and then \mathbb{E}^2 will be what we call the exterior bidisk.

We now restate and prove Theorem 1.5.

Theorem 1.5. *Let $\phi : \mathbb{D}^2 \rightarrow \mathcal{B}_1$ be a matrix valued inner function. Let X be an open subset of \mathbb{T}^2 and let*

$$\begin{aligned} X_1 &= \{x_1 \in \mathbb{T} : \exists x_2 \in \mathbb{T} \text{ with } (x_1, x_2) \in X\} \\ X_2 &= \{x_2 \in \mathbb{T} : \exists x_1 \in \mathbb{T} \text{ with } (x_1, x_2) \in X\} \\ S &= \{1/\bar{z} : \det \phi(z) = 0\}. \end{aligned}$$

Then the following are equivalent:

- (i) *The function ϕ extends continuously to X .*
- (ii) *For some pair (A^1, A^2) of Agler kernels of ϕ , the elements of $\mathcal{H}(A^1)$, $\mathcal{H}(A^2)$ extend continuously to X .*
- (iii) *There is a domain Ω containing*

$$\mathbb{D}^2 \cup X \cup (X_1 \times \mathbb{D}) \cup (\mathbb{D} \times X_2) \cup (\mathbb{E}^2 \setminus S)$$

on which ϕ and the elements of $\mathcal{K}_\phi, \mathcal{K}_\phi^1, \mathcal{K}_\phi^2$ extend to be analytic (and meromorphic on $\Omega \cup S$). Point evaluation in Ω is bounded in these spaces, and therefore, the kernels G, F^1, F^2, E^1, E^2 and all Agler kernels (A^1, A^2) of ϕ extend to be sesqui-analytic on $\Omega \times \Omega$.

We will prove $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$. As most analysis lies in proving $(i) \Rightarrow (iii)$, we consider that implication first.

Claim 1. ϕ extends to be analytic in some domain Ω .

Proof. Suppose that X is an open subset of \mathbb{T}^2 and ϕ extends to be continuous on $\mathbb{D}^2 \cup X$. Then, ϕ is invertible in a neighborhood $W^+ \subset \mathbb{D}^2$ with $X \subset \text{closure}(W^+)$ (since ϕ is a unitary almost everywhere on \mathbb{T}^2). The following

$$(6.1) \quad \phi(z) = (\phi(1/\bar{z})^*)^{-1}$$

gives a definition of ϕ on $W^- := \{1/\bar{z} : z \in W^+\}$. The extended ϕ is holomorphic on $W^+ \cup W^-$ and continuous on $W^+ \cup X \cup W^-$ since ϕ is unitary valued on X . By the continuous edge-of-the-wedge theorem (Theorem A of Rudin [30]), there is a domain Ω_0 containing $W^+ \cup X \cup W^-$, which depends only on X, W^\pm , on which ϕ extends to be holomorphic. Moreover, ϕ is already holomorphic on \mathbb{D}^2 , meromorphic in \mathbb{E}^2 , and holomorphic away from the set S using the definition (6.1).

We can extend this domain further using a result in Rudin [29] (Theorem 4.9.1, which we provide as Proposition 6.2 below). It says, roughly, that if a holomorphic function f on \mathbb{D}^2 extends analytically to a neighborhood N_x of some $x = (x_1, x_2) \in \mathbb{T}^2$, then f extends analytically to an open set containing $\{x_1\} \times \mathbb{D}$ and $\mathbb{D} \times \{x_2\}$. As the edge-of-the-wedge theorem guarantees ϕ extends to a neighborhood N_x of each $x \in X$,

Proposition 6.2 implies ϕ extends analytically to an open set U containing $(X_1 \times \mathbb{D}) \cup (\mathbb{D} \times X_2)$, and the open set depends only on the $\{N_x\}_{x \in X}$. This detail is contained in the *proof* of Proposition 6.2. \square

Claim 2. Elements of $\mathcal{K}_\phi, \mathcal{K}_\phi^1, \mathcal{K}_\phi^2$ are analytic in Ω .

Proof. Consider now $f \in \mathcal{K}_\phi$; the proof is similar for the other subspaces. Since $\phi^* f \in L^2_{--}$, we may write $f = \phi \overline{Z_1 Z_2} g$ for some $g \in H^2$. This allows us to define f analytically outside of \mathbb{D}^2 as follows:

$$f(z) = \frac{1}{z_1 z_2} \phi(z) \overline{g(1/\bar{z})}$$

for $z \in \mathbb{E}^2 \setminus S$. Note f is meromorphic in \mathbb{E}^2 . With this definition, for any compact subset $X_0 \subset X$ and $z \in X_0$

$$\lim_{r \searrow 1} f(rz) = \phi(z) \bar{z}_1 \bar{z}_2 \overline{g(\bar{z})} = f(z)$$

in $L^2(X_0)$ since as $r \searrow 1$

$$g(1/r\bar{z}) \rightarrow g(\bar{z})$$

in $L^2(\mathbb{T}^2)$, while $\phi(rz) \rightarrow \phi(z)$ uniformly for $z \in X_0$ by the assumed continuity.

On the other hand, for $r \nearrow 1$,

$$f_r(z) = f(rz) \rightarrow f(z)$$

in $L^2(\mathbb{T}^2)$. Therefore, f_r possesses two-sided limits in $L^2(X_0)$ (i.e. for $r \searrow 1$ and $r \nearrow 1$) for any compact subset $X_0 \subset X$.

The distributional edge-of-the-wedge theorem (Theorem B of Rudin [30]) now applies. It requires that

$$\lim_{r \rightarrow 1} \int_X f_r(z) \psi(z) d\sigma(z)$$

exist for every $\psi \in C_c^\infty(X)$. The convergence of f_r to f in $L^2(X_0)$ on either side of any compact $X_0 \subset X$ is more than enough for this. The conclusion of the edge-of-the-wedge theorem is that f has a holomorphic extension to a domain Ω_0 containing $W^+ \cup X \cup W^-$. An important part of the theorem is that the domain depends only on W^\pm, X .

Then, for each $x \in X$, f extends analytically to a neighborhood N_x of x , and so Proposition 6.2 implies that f extends analytically to an open set U containing $X_1 \times \mathbb{D}$ and $\mathbb{D} \times X_2$. Again, from the proof of the theorem, it is clear that U depends only on the $\{N_x\}_{x \in X}$, which in turn depended only on W^\pm, X .

As f is already holomorphic in $\mathbb{D}^2 \cup (\mathbb{E}^2 \setminus S)$ we may conclude that every $f \in \mathcal{K}_\phi$ is holomorphic in an open set

$$\Omega = \Omega_0 \cup U \cup \mathbb{D}^2 \cup (\mathbb{E}^2 \setminus S)$$

and meromorphic in $\Omega' = \Omega_0 \cup U \cup \mathbb{D}^2 \cup \mathbb{E}^2$. \square

Claim 3. Points of Ω are bounded point evaluations for $\mathcal{K}_\phi, \mathcal{K}_\phi^1, \mathcal{K}_\phi^2$.

Proof. Again, we consider only \mathcal{K}_ϕ . Let B be the set of bounded point evaluations of \mathcal{K}_ϕ in Ω . It is clear that $\mathbb{D}^2 \subset B$ since points of \mathbb{D}^2 are bounded point evaluations of all of H^2 . Also, $\mathbb{E}^2 \setminus S \subset B$ by the definition of exterior values of functions in \mathcal{K}_ϕ . As a first step, we show that B is relatively closed in Ω and this will in particular show that $X \subset B$.

Suppose $\{w^j\} \subset B$ and $w^j \rightarrow w \in \Omega$. For each $f \in \mathcal{K}_\phi$,

$$\sup\{|f(w^j)| : j \geq 0\} < \infty.$$

By the uniform boundedness principle, there is a constant M such that

$$|f(w^j)| = |\langle f, G_{w^j} \rangle_{\mathcal{K}_\phi}| \leq M \|f\|_{\mathcal{K}_\phi}.$$

for all $f \in \mathcal{K}_\phi$ and $j \geq 0$. As each f is holomorphic in Ω , $f(w^j) \rightarrow f(w)$ and so

$$|f(w)| \leq M \|f\|_{\mathcal{K}_\phi}.$$

Hence, $w \in B$ and B is a relatively closed subset of Ω .

To show B contains Ω_0 we need to refer to the local construction of Ω_0 as in the continuous edge-of-the-wedge theorem as proved in Rudin [30]. Refer to Proposition 6.1 below. Modulo rescaling and a change of coordinates, the main point is that around any point $x \in X$, any $f \in \mathcal{K}_\phi$ is extended to a neighborhood N_x of x in \mathbb{C}^2 via an integral formula which only depends on the values of f in a compact subset $K \subset W^+ \cup X \cup W^-$. Now, every $f \in \mathcal{K}_\phi$ is analytic in a neighborhood of such a K and so for all $f \in \mathcal{K}_\phi$

$$\sup\{|f(w)| : w \in K\} < \infty.$$

By the uniform boundedness principle, there is a constant M such that for all $w \in K$ and $f \in \mathcal{K}_\phi$

$$|f(w)| \leq M \|f\|_{\mathcal{K}_\phi}.$$

Because of this, the values of any f in N_x are controlled by f 's values in K and hence by M and $\|f\|_{\mathcal{K}_\phi}$. Thus, the points of Ω_0 (as constructed in the proof of the edge-of-the-wedge theorem) are bounded point evaluations of \mathcal{K}_ϕ .

Now consider the points of U , the set guaranteed by Proposition 6.2. The set U is constructed as a union of neighborhoods of the points in $X_1 \times \mathbb{D}$ and $\mathbb{D} \times X_2$ as follows:

$$U = \bigcup_{z \in X_1 \times \mathbb{D}} N_z \cup \bigcup_{w \in \mathbb{D} \times X_2} N_w.$$

Specifically, fix $z = (x_1, z_2) \in X_1 \times \mathbb{D}$. Then, there is an x_2 such that $(x_1, x_2) \in X$ and a neighborhood N_x of x (guaranteed by the edge-of-the-wedge theorem) such that each $f \in \mathcal{K}_\phi$ extends analytically to N_x . Then, Proposition 6.2 guarantees a neighborhood N_z of z to which each f extends analytically. It follows by the construction in the proof that there is a compact set K contained in $\mathbb{D}^2 \cup N_x$ such that for all $z_0 \in N_z$ and $f \in \mathcal{K}_\phi$

$$|f(z_0)| \leq \sup_{w \in K} |f(w)|.$$

We can again use the uniform boundedness principle to conclude that the points in N_z are also bounded point evaluations of \mathcal{K}_ϕ . Note that Ω is constructed in the proof essentially as

$$\mathbb{D}^2 \cup (\mathbb{E}^2 \setminus S) \cup \bigcup_{x \in X} N_x \cup \bigcup_{z \in X_1 \times \mathbb{D}} N_z \cup \bigcup_{w \in \mathbb{D} \times X_2} N_w,$$

and so we have proven that the points of Ω are bounded point evaluations of \mathcal{K}_ϕ .

Finally, the reproducing kernel $G(z, w)$ can now be extended to be sesqui-analytic in Ω . Similarly, the reproducing kernels of \mathcal{K}_ϕ^1 and \mathcal{K}_ϕ^2 can be extended to $\Omega \times \Omega$, which implies F^1, F^2, E^1, E^2 extend. Let (A^1, A^2) be Agler kernels of ϕ . By Corollary 5.3, the points of Ω are bounded point evaluations of $\mathcal{H}(A^1)$ and $\mathcal{H}(A^2)$, and hence, (A^1, A^2) extend to be sesqui-analytic in Ω as well. \square

That concludes the proof of $(i) \Rightarrow (iii)$, and it is immediate that $(iii) \Rightarrow (ii)$. Now consider $(ii) \Rightarrow (i)$:

Proof. Let (A^1, A^2) be Agler kernels of ϕ such that the elements of $\mathcal{H}(A^1)$ and $\mathcal{H}(A^2)$ extend continuously to X . By definition,

$$1 - \phi(z)\phi(w)^* = (1 - z_1\bar{w}_1)A^2(z, w) + (1 - z_2\bar{w}_2)A^1(z, w),$$

for all $z, w \in \mathbb{D}^2$. Since ϕ is an isometry almost everywhere on \mathbb{T}^2 , we can choose $w \in \mathbb{D}^2$ such that $\phi(w)$ is invertible. (Recall that we are assuming \mathcal{V} is finite dimensional, so that ϕ converges to its boundary values radially almost everywhere.) As $A_w^1\nu \in \mathcal{H}(A^1)$ and $A_w^2\nu \in \mathcal{H}(A^2)$ both extend to be continuous on X for all $\nu \in \mathcal{V}$, so does

$$\phi(z) = (-(1 - z_1\bar{w}_1)A_w^2(z) - (1 - z_2\bar{w}_2)A_w^1(z) + 1) (\phi(w)^*)^{-1}.$$

□

When showing (i) \Rightarrow (iii), we made reference to the following construction of Rudin. Notice in particular that the integral formula for F depends only on the values of f on the “wedge.”

Proposition 6.1. [30, pg 10] *Let E and V be open cubes in \mathbb{R}^2 defined as follows:*

$$\begin{aligned} E &= \{x : -6 < x_j < 6 \text{ for } 1 \leq j \leq 2\} \\ V &= \{y : 0 < y_j < 6 \text{ for } 1 \leq j \leq 2\}. \end{aligned}$$

Define $R^+ := E + iV$ and $R^- := E - iV$. Assume f is a function continuous on $R^+ \cup E \cup R^-$ and holomorphic on $R^+ \cup R^-$. Then, there exists a function F holomorphic on \mathbb{D}^2 with $F \equiv f$ on $\mathbb{D}^2 \cap (R^+ \cup E \cup R^-)$. Define

$$\psi(s, t) := \frac{s + \frac{t}{c}}{1 + cst}, \quad \text{where } c = \sqrt{2} - 1.$$

Then F is given by

$$F(\lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi(\lambda_1, e^{i\theta}), \psi(\lambda_2, e^{i\theta})) d\theta \quad \text{for } \lambda \in \mathbb{D}^2,$$

and for each pair (λ, θ) , the point $(\psi(\lambda_1, e^{i\theta}), \psi(\lambda_2, e^{i\theta}))$ is in $R^+ \cup E \cup R^-$.

For convenience we recount the following definitions and proposition from Rudin [29, pg 97-99], which were used above.

A boundary point p of \mathbb{D}^2 is *regular point* for a holomorphic $f : \mathbb{D}^2 \rightarrow \mathbb{C}$ if there is a neighborhood N_p of p where $f|_{N_p \cap \mathbb{D}^2}$ extends to be holomorphic on N_p . Otherwise, p is a *singular point* of f .

Proposition 6.2. [29, Theorem 4.9.1 pg 98] *If f is holomorphic in \mathbb{D}^2 , $\beta \in \mathbb{D}$, and $(1, \beta)$ is a singular point of f , then $(1, \eta)$ is a singular point of f for every $\eta \in \mathbb{T}$.*

The contrapositive implies that if f is regular at $(1, 1)$, then f is regular at $(1, \beta)$ for each $\beta \in \mathbb{D}$. It can be seen from the proof in [29] that if f is holomorphic in a neighborhood N of $(1, 1)$, then for each $\beta \in \mathbb{D}$ there is a compact set $K \subset \mathbb{D}^2 \cup N$ (depending only on N , β and not f) such that $|f(1, \beta)| \leq \max_K |f|$. In other words, $(1, \beta)$ is in the holomorphically convex hull of K in $\mathbb{D}^2 \cup N$.

7. THEOREM 1.6 ON RESTRICTION MAPS

Theorem 1.6. *Let ϕ be a finite dimensional matrix valued inner function on \mathbb{D}^2 . For almost every $t \in \mathbb{T}$, the map*

$$f \mapsto f(t, \cdot)$$

embeds $\mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi$ and $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ isometrically into $H^2(\mathbb{T}) \ominus \phi(t, \cdot) H^2(\mathbb{T})$.

Proof. The proof is the same for $\mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi$ and $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$. The key facts we use are that both spaces are contained in \mathcal{H}_ϕ^1 and since

$$\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi = (\mathcal{H}_\phi^1 \ominus \mathcal{K}_\phi) \ominus Z_1(\mathcal{H}_\phi^1 \ominus \mathcal{K}_\phi)$$

and

$$\mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi = \mathcal{H}_\phi^1 \ominus Z_1 \mathcal{H}_\phi^1,$$

both of these spaces are orthogonal to their translates by Z_1 .

We provide the proof now for only $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$. By the above observations, for any $f, g \in \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi = (\mathcal{H}_\phi^1 \ominus \mathcal{K}_\phi) \ominus Z_1(\mathcal{H}_\phi^1 \ominus \mathcal{K}_\phi)$, we have that $f \perp Z_1^j g$ for all $j \in \mathbb{Z}$ except $j = 0$. Therefore, for $j \neq 0$

$$0 = \int_{\mathbb{T}} z_1^j \int_{\mathbb{T}} \langle f(z), g(z) \rangle_{\mathcal{V}} d\sigma(z_2) d\sigma(z_1),$$

which implies that

$$\int_{\mathbb{T}} \langle f(z_1, z_2), g(z_1, z_2) \rangle_{\mathcal{V}} d\sigma(z_2) = \langle f, g \rangle$$

for almost every $z_1 \in \mathbb{T}$.

This shows the restriction map $f \mapsto f(t, \cdot)$ is an isometry from $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ to $H^2(\mathbb{T})$ for almost every $t \in \mathbb{T}$. (By separability, we can show that given a countable dense set \mathcal{D} in $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$, for almost every t , every $f \in \mathcal{D}$ possesses slices $f(t, \cdot) \in L^2(\mathbb{T})$. Since we will have an isometry on this dense set, it will extend to be isometric on the whole space.)

Now, $\phi(t, \cdot)$ is inner for almost every $t \in \mathbb{T}$, and since $f \in \mathcal{H}_\phi^1$ implies $f(t, \cdot) \in H^2(\mathbb{T})$ and $\phi(t, \cdot)^* f(t, \cdot) \in L_-^2(\mathbb{T})$ for almost every t , we see that for $f \in \mathcal{H}_\phi^1$

$$f(t, \cdot) \in H^2(\mathbb{T}) \ominus \phi(t, \cdot) H^2(\mathbb{T})$$

for almost every $t \in \mathbb{T}$. (Again, we could argue using separability that we are only taking “almost every t ” a countable number of times.)

Therefore, for almost every $t \in \mathbb{T}$, $f \mapsto f(t, \cdot)$ is an isometry from $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ into $H^2(\mathbb{T}) \ominus \phi(t, \cdot) H^2(\mathbb{T})$. □

An obvious question is then:

Question 7.1. *Is the restriction map above onto for almost every t ?*

We have been unable to resolve this but having some regularity on the boundary allows us to prove a partial result.

Proposition 7.2. *If ϕ extends continuously to a rectangle $X = X_1 \times X_2 \subset \mathbb{T}^2$, then the restriction map*

$$f \mapsto f(t, \cdot)$$

embeds $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ and $\mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi$ isometrically onto $H^2(\mathbb{T}) \ominus \phi(t, \cdot) H^2(\mathbb{T})$ for almost every $t \in X_1$.

Proof. As before we treat the case $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$. By the regularity results, ϕ and elements of \mathcal{K}_ϕ^1 extend analytically to a domain Ω containing $\mathbb{D}^2, X, (X_1 \times \mathbb{D})$. In addition, the reproducing kernels F^1 and E^2 are sesqui-analytic on $\Omega \times \Omega$. For $t \in X_1, \zeta, \eta \in \mathbb{D}$, we can use Theorem 1.3 to conclude

$$\frac{1 - \phi(t, \zeta) \phi(t, \eta)^*}{1 - \zeta \bar{\eta}} = F^1((t, \zeta), (t, \eta)).$$

Therefore, for $\eta \in \mathbb{D}, \nu \in \mathcal{V}$,

$$F_{(t, \eta)}^1 \nu \mapsto \frac{1 - \phi(t, \cdot) \phi(t, \eta)^*}{1 - (\cdot) \bar{\eta}} \nu$$

under the restriction map for $t \in X_1$. Since $\phi(t, \cdot)$ is inner for almost every t , this shows that the image of $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ under this restriction map is a dense subset of $H^2(\mathbb{T}) \ominus \phi(t, \cdot) H^2(\mathbb{T})$ (namely the dense subset of the span of reproducing kernels) for almost every $t \in X_1$. Since the restriction map is an isometry for almost every t , it must therefore be a unitary for almost every $t \in X_1$. \square

Example 7.3. The example $\phi(z) = \frac{2z_1 z_2 - z_1 - z_2}{2 - z_1 - z_2}$ shows that we cannot have an isometry for *every* t in the restriction map of Theorem 1.6. The simple reason is that $H^2(\mathbb{T}) \ominus \phi(t, \cdot) H^2(\mathbb{T})$ has dimension 1 for all t except $t = 1$ where it has dimension 0, since $\phi(1, z_2) = -1$. \diamond

8. A TECHNICAL FACT

In the next sections we study rational inner functions. The following technical fact is essential. A similar fact was needed in [22].

Proposition 8.1. *Suppose $p \in \mathbb{C}[z]$ has no zeros in \mathbb{D}^2 , $f \in H^2$, and $f/p \in L^2$. Then, $f/p \in H^2$.*

Proof. By Fubini's theorem, $f(\cdot, z_2) \in H^2(\mathbb{T})$ for a.e. $z_2 \in \mathbb{T}$; the same holds for $p(\cdot, z_2)$. Recall that for a polynomial to be *outer*, in the sense of Hardy spaces in the disk, it is necessary and sufficient that it have no zeros in \mathbb{D} . By Lemma 8.2 below, $p(\cdot, z_2)$ is outer for all but finitely many $z_2 \in \mathbb{T}$ since p has no zeros on \mathbb{D}^2 , and therefore both $f(\cdot, z_2)$ and $1/p(\cdot, z_2)$ are in the Smirnov class N^+ for a.e. $z_2 \in \mathbb{T}$. As N^+ is an algebra, $f(\cdot, z_2)/p(\cdot, z_2)$ is in N^+ for a.e. $z_2 \in \mathbb{T}$. Since $N^+ \cap L^2(\mathbb{T}) = H^2(\mathbb{T})$ (see [14]), $f(\cdot, z_2)/p(\cdot, z_2) \in H^2(\mathbb{T})$ for a.e. $z_2 \in \mathbb{T}$. This implies $f/p \perp L^2_{-\bullet}$. A similar argument shows $f/p \perp L^2_{-\bullet}$. Therefore, $f/p \in H^2$. \square

Lemma 8.2. *If $p \in \mathbb{C}[z_1, z_2] = \mathbb{C}[z]$ has no zeros on \mathbb{D}^2 , then for all $z_2 \in \mathbb{T}$ with at most a finite number of exceptions, $p(\cdot, z_2)$ has no zeros on \mathbb{D} .*

Proof. For $0 < r < 1$ and $\zeta \in \mathbb{T}$, $p(\cdot, r\zeta)$ has no zeros in \mathbb{D} . By Hurwitz's theorem, it follows that $p(\cdot, \zeta)$ is either identically zero or has no zeros in \mathbb{D} . If $p(\cdot, \zeta)$ is identically zero, p must have $z_2 - \zeta$ as a factor. As $p \in \mathbb{C}[z]$ can only have finitely many factors of this form, the claim follows. \square

9. MATRIX RATIONAL INNER FUNCTIONS AND THEOREM 1.7

Suppose ϕ is a rational matrix inner function. Then we write ϕ as

$$\phi(z) = \frac{Q(z)}{p(z)},$$

where $p \in \mathbb{C}[z]$ is the least common multiple of the denominators of the entries of ϕ after each entry is put into reduced form and has no zeros in \mathbb{D}^2 , and $Q \in \mathbb{C}^{N \times N}[z]$ is a matrix polynomial satisfying

$$Q^*Q = |p|^2I = Q^t\bar{Q} \text{ on } \mathbb{T}^2.$$

Lemma 9.1. *With $\phi = Q/p$ as above, p has finitely many zeros on \mathbb{T}^2 .*

Proof. For Q/p to be holomorphic, it is necessary that p have no zeros in \mathbb{D}^2 . Every polynomial with no zeros in \mathbb{D}^2 may be factored into two such polynomials $p = p_1p_2$ where p_1 has finitely many zeros on \mathbb{T}^2 , and each irreducible factor of p_2 has infinitely many zeros on \mathbb{T}^2 . We allow either factor to be a constant. (This is the atoral-toral factorization of Agler-McCarthy-Stankus [3].) Our claim is that p_2 is a constant. If p_2 has some nontrivial irreducible factor f , then since $Q^*Q = |p|^2I$, every entry of Q vanishes on the zero set of f and hence every entry is divisible by f . (Two bivariate polynomials with infinitely many common zeros must have factor in common.) This contradicts the fact that p is the least common multiple of the denominators of ϕ . \square

Let $d = (d_1, d_2)$ be the maximal degree of Q . Then, $\tilde{Q}(z) = z^d Q(1/\bar{z})^*$ is a matrix polynomial and $\tilde{p}(z) = z^d \overline{p(1/\bar{z})}$ is a polynomial. Set

$$\tilde{\phi} = \frac{\tilde{Q}}{\tilde{p}},$$

which is inner since $\tilde{Q}\tilde{Q}^* = Q^*Q = |p|^2 I$ on \mathbb{T}^2 . Notice also that $\phi\tilde{\phi} = \frac{\tilde{p}}{p}I$. Define

$$\begin{aligned}\mathcal{P}_{d,p}^0 &= \{q/p \in H^2 : q \in \mathbb{C}^N[z], \deg q \leq (d_1 - 1, d_2 - 1)\} \\ \mathcal{P}_{d,p}^1 &= \{q/p \in H^2 : q \in \mathbb{C}^N[z], \deg q \leq (d_1, d_2 - 1)\} \\ \mathcal{P}_{d,p}^2 &= \{q/p \in H^2 : q \in \mathbb{C}^N[z], \deg q \leq (d_1 - 1, d_2)\}.\end{aligned}$$

Note that these spaces depend on the degree of Q (and not necessarily p).

We repeat Theorem 1.7 here for convenience.

Theorem 1.7. *If ϕ is a rational matrix inner function, then*

$$\begin{aligned}\mathcal{K}_\phi &= \{f \in \mathcal{P}_{d,p}^0 : \tilde{\phi}f \in \mathcal{P}_{d,p}^0\} \\ \mathcal{K}_\phi^1 &= \{f \in \mathcal{P}_{d,p}^1 : \tilde{\phi}f \in \mathcal{P}_{d,p}^1\} \\ \mathcal{K}_\phi^2 &= \{f \in \mathcal{P}_{d,p}^2 : \tilde{\phi}f \in \mathcal{P}_{d,p}^2\}.\end{aligned}$$

Proof. We prove the theorem only for \mathcal{K}_ϕ ; the claims for $\mathcal{K}_\phi^1, \mathcal{K}_\phi^2$ are similar. Set $d' = (d_1 - 1, d_2 - 1)$ and $Z^{d'} = Z_1^{d_1-1} Z_2^{d_2-1}$. We frequently use the fact that $f \in H^2$ and $Z^{d'} \bar{f} \in H^2$ implies f is a polynomial of degree at most d' .

If $g \in \mathcal{K}_\phi$, then $g \in H^2$ and

$$g = \frac{Q}{p} \overline{Z_1 Z_2 h}$$

for some $h \in H^2$. Then,

$$\begin{aligned}pg &= Q \overline{Z_1 Z_2 h} \in H^2 \\ Z^{d'} \overline{pg} &= Z^d \bar{Q} h = \tilde{Q}^t h \in H^2\end{aligned}$$

shows $q = pg$ is a polynomial of degree at most d' ; i.e. $g \in \mathcal{P}_{d,p}^0$. This also shows $\tilde{q} = \tilde{Q}^t h$ is a polynomial of degree at most d' . Observe now that

$$Q^t \tilde{q} = Q^t \tilde{Q}^t h = \tilde{p} p h,$$

which implies that

$$\frac{Q^t \tilde{q}}{\tilde{p}} = p h \in H^2.$$

On the other hand,

$$Z^{d'} \overline{ph} = \frac{Q^* q}{\bar{Z}^d p} = \frac{\tilde{Q} q}{p} = \tilde{\phi} q \in H^2,$$

which shows ph is a polynomial of degree at most d' ; i.e. $h \in \mathcal{P}_{d,p}^0$. In addition, $\tilde{\phi} q$ is a polynomial of degree at most d' and therefore,

$$\tilde{\phi} \frac{q}{p} = \tilde{\phi} g \in \mathcal{P}_{d,p}^0.$$

Thus, we have proved

$$\mathcal{K}_\phi \subset \{f \in \mathcal{P}_{d,p}^0 : \tilde{\phi} f \in \mathcal{P}_{d,p}^0\}$$

and need to establish the opposite inclusion. Suppose $f \in \mathcal{P}_{d,p}^0$ and

$$(9.1) \quad \frac{\tilde{Q}}{p} f = \frac{r}{p}$$

with $r \in \mathbb{C}[z]$ and $\deg r \leq d'$. Set $\tilde{r} = Z^{d'} \bar{r}$. We must show $\phi^* f \in L_{--}^2$. Observe that $Z^d Q^* f = r$ implies $Q^* f = \frac{r}{Z_1 Z_2 \tilde{r}}$ and hence,

$$\phi^* f = \bar{Z}_1 \bar{Z}_2 \frac{\tilde{r}}{p}.$$

By (9.1), $r/p \in L^2$, which implies $\tilde{r}/p \in L^2$ and so by Proposition 8.1, \tilde{r}/p is in H^2 . Therefore, $\phi^* f \in L_{--}^2$ as desired. \square

By our definitions, in the scalar case, $\tilde{\phi} = \frac{p}{p} = 1$.

Corollary 9.2. *If $\phi = \tilde{p}/p$ is a scalar rational inner function, then*

$$\mathcal{K}_\phi = \mathcal{P}_{d,p}^0 \quad \mathcal{K}_\phi^1 = \mathcal{P}_{d,p}^1 \quad \mathcal{K}_\phi^2 = \mathcal{P}_{d,p}^2.$$

If p has no zeros on \mathbb{T}^2 , then

$$\mathcal{K}_\phi = \{q/p : \deg q \leq (d_1 - 1, d_2 - 1)\}$$

and similarly for $\mathcal{K}_\phi^1, \mathcal{K}_\phi^2$.

As an interesting aside, note that for scalar ϕ , $\mathcal{K}_\phi = \{q/p : \deg q \leq (d_1 - 1, d_2 - 1)\}$ if and only if $1/p \in L^2(\mathbb{T}^2)$ if and only if p has no zeros on \mathbb{T}^2 . The main thing to check is that $1/p \in L^2(\mathbb{T}^2)$ implies p has no zeros in \mathbb{T}^2 . Since p has only finitely many zeros on \mathbb{T}^2 , this is a local problem. So, let us assume $p(1, 1) = 0$ and prove

$$\iint_{[-\epsilon, \epsilon]^2} \frac{1}{|p(e^{i\theta_1}, e^{i\theta_2})|^2} d\theta_1 d\theta_2 = \infty.$$

In this case,

$$p(z) = \sum_{1 \leq |\alpha| \leq n} C_\alpha (1 - z_1)^{\alpha_1} (1 - z_2)^{\alpha_2},$$

where $|\alpha| = \alpha_1 + \alpha_2$, and then

$$|p(e^{i\theta_1}, e^{i\theta_2})|^2 \leq \text{const}(|1 - e^{i\theta_1}|^2 + |1 - e^{i\theta_2}|^2) \leq \text{const}(\theta_1^2 + \theta_2^2).$$

This shows

$$\iint_{[-\epsilon, \epsilon]^2} \frac{1}{|p(e^{i\theta_1}, e^{i\theta_2})|^2} d\theta_1 d\theta_2 \geq \text{const} \iint_{[-\epsilon, \epsilon]^2} \frac{1}{\theta_1^2 + \theta_2^2} d\theta_1 d\theta_2,$$

which diverges.

10. EXAMPLES

We use several examples to highlight differences between the situation when ϕ is scalar rational inner and the situation when ϕ is matrix rational inner.

First, if ϕ is scalar rational inner, continuous on $\overline{\mathbb{D}^2}$, and has a unique Agler decomposition, then ϕ is a function of one variable. (See [22].) This result fails when ϕ is matrix rational inner. Clearly, if $\phi(z)$ is diagonal with functions of one variable alone on the diagonal, then ϕ has a unique Agler decomposition. This still holds if we replace ϕ with $U\phi U^*$ where U is a constant unitary matrix.

However, those are not the only matrix rational inner functions with unique decompositions.

Example 10.1. Let

$$\phi(z) = \frac{1}{2} \begin{pmatrix} z_1(z_1 + z_2) & z_1(z_1 - z_2) \\ z_1 - z_2 & z_1 + z_2 \end{pmatrix}.$$

Then ϕ is matrix rational inner, and it is easy to show that ϕ is not of the form $UD(z)U^*$, where U is unitary and $D(z)$ is diagonal with entries that are functions of either z_1 or z_2 alone. We use Theorem 1.7 to calculate \mathcal{K}_ϕ . Let $f \in \mathcal{K}_\phi$. Then $\deg f \leq (1, 0)$, and we can write

$$f(z) = \begin{pmatrix} a_1 + b_1 z_1 \\ a_2 + b_2 z_1 \end{pmatrix},$$

for constants a_1, a_2, b_1, b_2 . An easy calculation gives

$$\tilde{\phi}(z) = \frac{1}{2} \begin{pmatrix} z_1 + z_2 & z_1(z_2 - z_1) \\ z_2 - z_1 & z_1(z_1 + z_2) \end{pmatrix}.$$

As $f \in \mathcal{K}_\phi$, we have

$$\tilde{\phi}(z)f = \frac{1}{2} \begin{pmatrix} z_1 + z_2 & z_1(z_2 - z_1) \\ z_2 - z_1 & z_1(z_1 + z_2) \end{pmatrix} \begin{pmatrix} a_1 + b_1 z_1 \\ a_2 + b_2 z_1 \end{pmatrix} = \begin{pmatrix} c_1 + d_1 z_1 \\ c_2 + d_2 z_2 \end{pmatrix},$$

where c_1, c_2, d_1, d_2 are constants. Thus,

$$\frac{1}{2}(z_1 + z_2)(a_1 + b_1 z_1) + \frac{1}{2}z_1(z_2 - z_1)(a_2 + b_2 z_1) = c_1 + d_1 z_1,$$

and an examination of the coefficients implies $a_1 = a_2 = b_1 = b_2 = 0$. Thus, $\mathcal{K}_\phi = \{0\}$, and it follows from Corollary 4.1 that ϕ has a unique Agler decomposition. The decomposition is given by

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} z_1(z_1 + z_2) & z_1(z_1 - z_2) \\ z_1 - z_2 & z_1 + z_2 \end{pmatrix} \begin{pmatrix} \bar{w}_1(\bar{w}_1 + \bar{w}_2) & \bar{w}_1 - \bar{w}_2 \\ \bar{w}_1(\bar{w}_1 - \bar{w}_2) & \bar{w}_1 + \bar{w}_2 \end{pmatrix} \\ &= (1 - z_1\bar{w}_1) \left(\frac{1}{2} \begin{pmatrix} 0 & z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \bar{w}_1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 \right) \\ &+ (1 - z_2\bar{w}_2) \frac{1}{2} \begin{pmatrix} 0 & -z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\bar{w}_1 & 1 \end{pmatrix}. \end{aligned}$$

◇

Question 10.2. *Can one characterize the regular matrix rational inner functions with unique Agler decompositions? What can be said in the non-regular case?*

Now we consider another difference between the matrix and scalar cases. For $\phi = Q/p$ rational inner, observe that $\deg Q$ is not necessarily the exact degree of every entry of Q . This discrepancy breaks down some of the structure seen in the scalar rational inner case. As shown in Corollary 9.2, for ϕ scalar rational inner, $\mathcal{P}_{d,p}^0 \subseteq \mathcal{H}_\phi$, and $\mathcal{H}_\phi \cap \mathcal{P}_{d,p}^0 = \mathcal{K}_\phi$. However, the following example illustrates that neither relation holds for an arbitrary rational matrix inner function.

Example 10.3. Let $\phi_1(z) = \frac{3z_1z_2 - z_1 - z_2}{3 - z_1 - z_2}$ and $\phi_2(z) = z_1^2z_2^2$ and define

$$\phi(z) = \begin{pmatrix} \phi_1(z) & 0 \\ 0 & \phi_2(z) \end{pmatrix}.$$

Then $p(z) = 3 - z_1 - z_2$, and we can rewrite ϕ as

$$\frac{Q(z)}{p(z)} = \frac{1}{3 - z_1 - z_2} \begin{pmatrix} 3z_1z_2 - z_1 - z_2 & 0 \\ 0 & z_1^2z_2^2(3 - z_1 - z_2) \end{pmatrix},$$

so that $\deg Q = (3, 3)$. Observe that

$$\mathcal{H}_\phi = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_i \in \mathcal{H}_{\phi_i} \text{ for } i = 1, 2 \right\},$$

and we can calculate \mathcal{H}_{ϕ_i} from Proposition 4.8 in [12] as follows:

$$\mathcal{H}_{\phi_1} = \left\{ \frac{f}{p} \in H^2(\mathbb{T}^2) : \hat{f}(j_1, j_2) = 0 \text{ for } j_1 > 0 \text{ and } j_2 > 0 \right\}$$

$$\mathcal{H}_{\phi_2} = \left\{ f \in H^2(\mathbb{T}^2) : \hat{f}(j_1, j_2) = 0 \text{ for } j_1 > 1 \text{ and } j_2 > 1 \right\}.$$

It is almost immediate that $\mathcal{P}_{d,p}^0 \not\subseteq \mathcal{H}_\phi$. Specifically, one can show

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} / p \in \mathcal{P}_{d,p}^0 \cap \mathcal{H}_\phi$$

if and only if each term in q_1 has degree zero in one variable and degree at most two in the other, and q_2 is of the form $p(z)r(z)$, where $\deg r \leq (1, 1)$. Thus, $\mathcal{P}_{d,p}^0 \cap \mathcal{H}_\phi \neq \mathcal{P}_{d,p}^0$.

Using Theorem 1.7, one can show \mathcal{K}_ϕ is the set

$$\mathcal{K}_\phi = \left\{ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} / p : q_1 \in \mathbb{C}, q_2 = pr, \text{ where } \deg r \leq (1, 1) \right\}.$$

Thus, $\mathcal{P}_{d,p}^0 \cap \mathcal{H}_\phi \not\subseteq \mathcal{K}_\phi$.

◇

Example 10.4. There are still interesting questions in the scalar case. First, observe that the set of Agler kernels (A^1, A^2) of a function ϕ is a convex set. Now, consider the space $C(\mathbb{D}^4) \times C(\mathbb{D}^4)$, the direct product of the space of continuous functions on \mathbb{D}^4 with itself, endowed with the topology of uniform convergence on compact sets. It is easy to show that the set of Agler kernels of ϕ is a compact subset of $C(\mathbb{D}^4) \times C(\mathbb{D}^4)$, i.e. that every sequence has a subsequence that converges to a pair of Agler kernels of ϕ . Then we can apply the Krein-Milman theorem to conclude that the set of Agler kernels of ϕ is the closed convex hull of its extreme points.

Agler kernels (A^1, A^2) are said to come from an orthogonal decomposition if $A^1/(1 - z_1\bar{w}_1), A^2/(1 - z_2\bar{w}_2)$ are reproducing kernels of closed subspaces of \mathcal{H}_ϕ . This is equivalent to G^1, G^2 in Theorem 1.4 being reproducing kernels of orthogonal closed subspaces of \mathcal{K}_ϕ whose direct sum is all of \mathcal{K}_ϕ . Agler kernels coming from an orthogonal decomposition are extreme points in the set of Agler kernels of ϕ . We shall use the basic example $\phi(z) = z_1^2 z_2$ to show that there can be other extreme points in the set of Agler kernels, and we raise the following question.

Question 10.5. *Given an inner ϕ , can one describe the extreme points in the set of all Agler kernels associated to ϕ ?*

For $\phi(z) = z_1^2 z_2$, $\mathcal{K}_\phi = \vee\{1, Z_1\}, \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi = \vee\{Z_1^2\}, \mathcal{K}_\phi^2 \ominus \mathcal{K}_\phi = \vee\{Z_2, Z_1 Z_2\}$, which imply that

$$G = 1 + z_1\bar{w}_1, \quad F^1 = z_1^2\bar{w}_1^2, \quad F^2 = z_2\bar{w}_2(1 + z_1\bar{w}_1).$$

Theorem 1.4 shows that the only way to construct Agler kernels (A^1, A^2) of ϕ is to choose positive kernels G^1, G^2 such that

$$z_1^2\bar{w}_1^2 + (1 - z_1\bar{w}_1)G^1 = A^1 \succcurlyeq 0$$

and

$$z_2\bar{w}_2(1 + z_1\bar{w}_1) + (1 - z_2\bar{w}_2)G^2 = A^2 \succcurlyeq 0.$$

Observe that A^1, A^2 only come from an orthogonal decomposition if additionally, G^1, G^2 are kernels of subspaces of \mathcal{K}_ϕ . It is easy to see that the only possible G^1, G^2 coming from such subspaces are

- (1) $G^1 = G, G^2 = 0$.
- (2) $G^1 = 0, G^2 = G$.
- (3) $G^1 = (a + bz_1)\overline{(a + bw_1)}, G^2 = (\bar{b} - \bar{a}z_1)\overline{(\bar{b} - \bar{a}w_1)}$, where $|a|^2 + |b|^2 = 1$.

The first two possibilities can indeed occur. The third possibility only occurs when $a = 0, |b| = 1$. To see this, note that in the third possibility

$$\frac{A^1}{1 - z_1\bar{w}_1} = G^1 + \frac{z_1^2\bar{w}_1^2}{1 - z_1\bar{w}_1}$$

must be the reproducing kernel of a subspace, call it S , of \mathcal{H}_ϕ , which is invariant under multiplication by Z_1 . So, if $G^1 = (a + bz_1)\overline{(a + bw_1)}$, then $a + bZ_1 \in S$, and so $aZ_1 + bZ_1^2 \in S$. Since $Z_1^2 \in \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$, we must have $aZ_1 \in S$. Therefore, if $a \neq 0$, then $Z_1 \in S$, which implies $1 \in S$. This puts us back in case (1) above. So, $a = 0$ and $|b| = 1$, which really means the only possibility is $G^1 = z_1\bar{w}_1, G^2 = 1$.

Therefore the only Agler kernels coming from orthogonal decompositions are

- (1) $A^1(z, w) = 1, \quad A^2(z, w) = z_2\bar{w}_2(1 + z_1\bar{w}_1)$
- (2) $A^1(z, w) = z_1^2\bar{w}_1^2, \quad A^2(z, w) = 1 + z_1\bar{w}_1$
- (3) $A^1(z, w) = z_1\bar{w}_1, \quad A^2(z, w) = 1 + z_1z_2\bar{w}_1\bar{w}_2$.

Convex combinations of these kernels are of the form

$$(10.1) \quad \begin{aligned} A^1(z, w) &= a + bz_1\bar{w}_1 + cz_1^2\bar{w}_1^2 \\ A^2(z, w) &= az_2\bar{w}_2(1 + z_1\bar{w}_1) + b(1 + z_1z_2\bar{w}_1\bar{w}_2) + c(1 + z_1\bar{w}_1), \end{aligned}$$

where $a + b + c = 1, a, b, c \geq 0$. On the other hand, the following is a pair of Agler kernels (A^1, A^2) which are not of this form (we evaluate on the diagonal to save space).

$$\begin{aligned} A^1(z, z) &= \frac{1}{4}|1 + z_1|^2 + \frac{1}{4}|z_1|^2|1 - z_1|^2 \\ &= |z_1^2|^2 + (1 - |z_1|^2)G^1(z, z), \end{aligned}$$

where $G^1(z, z) = \frac{1}{4}(|1 + z_1|^2 + 2|z_1|^2)$, and

$$\begin{aligned} A^2(z, z) &= \frac{1}{2} + \frac{1}{2}|z_1z_2|^2 + \frac{1}{4}|1 - z_1|^2 + \frac{1}{4}|z_2|^2|1 + z_1|^2 \\ &= |z_2|^2(1 + |z_1|^2) + (1 - |z_2|^2)G^2(z, z), \end{aligned}$$

where $G^2(z, z) = \frac{1}{4}(2 + |1 - z_1|^2)$. The pair (A^1, A^2) is not of the form (10.1) since A^1 , for instance, contains a z_1 term, but there is no such term in (10.1). \diamond

11. REVIEW OF DIMENSIONS IN ONE VARIABLE

Before we move on to determine the dimensions of certain canonical subspaces in two variables, it helps to review what happens in the matrix case in one variable.

If ϕ is a matrix rational inner function of one variable, the space $\mathcal{H}_\phi = H^2(\mathbb{T}) \ominus \phi H^2(\mathbb{T})$ has dimension determined by the degree of $\det \phi$. Specifically, $\det \phi$ is a finite Blaschke product, and the dimension of \mathcal{H}_ϕ is the number of factors in the Blaschke product:

$$(11.1) \quad \dim H^2(\mathbb{T}) \ominus \phi H^2(\mathbb{T}) = \deg \det \phi.$$

This is known as the Smith-McMillan degree.

To prove this, we need the Smith Normal form (see [19, Section 7.4]). Writing $\phi = Q/p$, there exist matrix polynomials S, T with matrix polynomial inverses such that

$$S^{-1}QT^{-1} = D = \text{diag}(D_1, D_2, \dots, D_N)$$

is a diagonal matrix polynomial where D_i divides D_{i+1} .

Then,

$$\phi H^2 = \frac{1}{p}SDTH^2 = \frac{1}{p}SDH^2.$$

Now, $H^2 \ominus \phi H^2$ is isomorphic as a vector space to the quotient $H^2/\phi H^2$, which is in turn isomorphic to

$$S^{-1}H^2/S^{-1}\phi H^2 = H^2/\frac{D}{p}H^2.$$

Since D is diagonal, this space is an algebraic direct sum of scalar spaces $H^2/\frac{D_j}{p}H^2$. Each factor has dimension given by the number of zeros of D_j which lie in the unit disk. Since $\det S = s_0$, $\det T = t_0$ are nonzero constants (since these matrix polynomials have matrix polynomial inverses), and since

$$\det \phi = s_0 t_0 \frac{\prod D_j}{p^N}$$

is a Blaschke product, the total dimension will be the number of zeros of this Blaschke product, counting multiplicities.

12. THEOREM 1.8 ON DIMENSIONS OF CANONICAL SUBSPACES

We now assume $\phi = Q/p$ is an $N \times N$ matrix valued rational inner function on \mathbb{D}^2 as in Section 9. The scalar function $\det \phi = \frac{1}{p^N} \det Q$ is a rational inner function on \mathbb{D}^2 . It therefore has a representation as

$$\det \phi = \frac{\tilde{g}}{g},$$

where g is a polynomial with no zeros on \mathbb{D}^2 and no factors in common with \tilde{g} , and $\tilde{g}(z) = z_1^{M_1} z_2^{M_2} \overline{g(1/\bar{z}_1, 1/\bar{z}_2)}$ for some integers M_1, M_2 (see Rudin [29] Section 5.2). We necessarily have that g divides p^N , which means g has finitely many zeros on \mathbb{T}^2 .

Theorem 1.8.

$$\begin{aligned} \dim \mathcal{K}_\phi^1 \ominus Z_1 \mathcal{K}_\phi &= \dim \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi = \deg_2 \tilde{g} \\ \dim \mathcal{K}_\phi^2 \ominus Z_2 \mathcal{K}_\phi &= \dim \mathcal{K}_\phi^2 \ominus \mathcal{K}_\phi = \deg_1 \tilde{g}. \end{aligned}$$

The notation $\deg_j q$ refers to the degree of $q(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ in the variable z_j .

Proof. As the argument is the same for $\mathcal{K}_\phi^j \ominus Z_j \mathcal{K}_\phi$ and $\mathcal{K}_\phi^j \ominus \mathcal{K}_\phi$, we only address $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$.

There are only finitely many $t \in \mathbb{T}$ such that p has a zero on the line $\{t\} \times \overline{\mathbb{D}}$ by Lemmas 8.2 and 9.1. If we choose t such that p has no zeros on $\{t\} \times \overline{\mathbb{D}}$, then ϕ will be analytic in a neighborhood of $\{t\} \times \mathbb{T}$. Perturbing t if necessary, by Proposition 7.2, the map

$$\frac{f}{p} \mapsto \frac{f(t, \cdot)}{p(t, \cdot)}$$

will map $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ onto $H^2(\mathbb{T}) \ominus \phi(t, \cdot)H^2(\mathbb{T})$ isometrically. Hence,

$$(12.1) \quad \dim \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi = \dim H^2(\mathbb{T}) \ominus \phi(t, \cdot)H^2(\mathbb{T}).$$

As g has no zeros on the line $\{t\} \times \overline{\mathbb{D}}$ and no zeros in \mathbb{D}^2 , $\tilde{g}(t, \cdot)/g(t, \cdot)$ is a Blaschke product of degree $\deg \tilde{g}(t, \cdot)$. (The degree could have been less if $g(t, \cdot)$ had a zero on \mathbb{T} .) Further,

$$(12.2) \quad \deg \tilde{g}(t, \cdot) = \deg_2 \tilde{g}.$$

To see this, write $M_2 = \deg_2 \tilde{g}$ and

$$g(z_1, z_2) = \sum_{j=0}^{M_2} g_j(z_1) z_2^j.$$

We see that

$$\tilde{g}(z_1, z_2) = \sum_{j=0}^{M_2} \tilde{g}_{M_2-j}(z_1) z_2^j,$$

where we perform “reflection” of the one variable polynomials at the appropriate degree M_1 . The top coefficient is $\tilde{g}_0(z_1)$, which does not vanish for $z_1 = t$, else $g(t, 0) = g_0(t)$ would vanish (which means g would vanish on $\{t\} \times \overline{\mathbb{D}}$, where it does not). Hence, $\tilde{g}(t, z_2)$ has degree precisely M_2 in z_2 .

Combining (12.2), (11.1), and (12.1), we have

$$\deg_2 \tilde{g} = \deg \tilde{g}(t, \cdot) = \dim H^2(\mathbb{T}) \ominus \phi(t, \cdot) H^2(\mathbb{T}) = \dim \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi,$$

as desired. □

These dimension results also hold for reproducing kernel Hilbert spaces associated to more general Agler decompositions.

Remark 12.1. Assume (A^1, A^2) are Agler kernels of ϕ such that the reproducing kernel Hilbert spaces with kernels

$$\frac{A^1(z, w)}{1 - z_1 \bar{w}_1} \text{ and } \frac{A^2(z, w)}{1 - z_2 \bar{w}_2}$$

are closed subspaces of H^2 . Then, $\mathcal{H}(A^1)$ and $\mathcal{H}(A^2)$ are orthogonal to their translates by Z_1 and Z_2 respectively. Moreover, $\mathcal{H}(A^1) \subseteq \mathcal{H}_\phi^1$ and $\mathcal{H}(A^2) \subseteq \mathcal{H}_\phi^2$. The subspaces discussed in Theorem 1.8 are clearly special cases of these general reproducing kernel Hilbert spaces, and the arguments in Theorem 1.6, Proposition 7.2, and Theorem 1.8 are valid for these more general $\mathcal{H}(A^1)$ and $\mathcal{H}(A^2)$. Specifically,

$$\dim \mathcal{H}(A^1) = \deg_2 \tilde{g}$$

$$\dim \mathcal{H}(A^2) = \deg_1 \tilde{g}.$$

On the other hand, for *general* Agler kernels the best that can be said is

$$\dim \mathcal{K}_\phi^1 \geq \dim \mathcal{H}(A^1) \geq \deg_2 \tilde{g}$$

$$\dim \mathcal{K}_\phi^2 \geq \dim \mathcal{H}(A^2) \geq \deg_1 \tilde{g}.$$

The upper bounds follow from Corollary 5.3 while the lower bounds are the content of Corollary 1.9, which we now prove.

Proof of Corollary 1.9. By Theorem 1.4 there exists a positive kernel $G^1 \preceq G$ so that

$$A^1(z, w) = F^1(z, w) + (1 - z_1 \bar{w}_1) G^1(z, w).$$

By Proposition 7.2 we may choose $t \in \mathbb{T}$ such that the restriction map $f \mapsto f(t, \cdot)$ maps $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ one-to-one and onto $H^2(\mathbb{T}) \ominus \phi(t, \cdot)H^2(\mathbb{T})$. This shows the kernels $F_{(t,\eta)}^1 v$, where η varies over \mathbb{D} and $v \in \mathcal{V}$, are dense in $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$ since any f orthogonal to all such kernels would vanish on the set $\{(t, \eta) : \eta \in \mathbb{D}\}$. Such an f would then map to zero under the restriction map, contradicting the fact that it is one-to-one and onto.

Finally,

$$A^1((t, z_2), (t, w_2)) = F^1((t, z_2), (t, w_2))$$

and this shows $\dim \mathcal{H}(A^1)$ is at least the dimension of the space spanned by $F_{(t,\eta)}^1(t, \cdot)v$ where $\eta \in \mathbb{D}$, $v \in \mathcal{V}$. This is the same as the dimension of the space spanned by $F_{(t,\eta)}^1 v$ where $\eta \in \mathbb{D}$, $v \in \mathcal{V}$, since the restriction map is bijective. Therefore,

$$\dim \mathcal{H}(A^1) \geq \dim \mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi = \deg_2 \tilde{g}.$$

The proof for A^2 is similar. □

Proof of Corollary 1.11. Given our Agler decomposition

$$1 - \phi(z)\phi(w)^* = (1 - z_1\bar{w}_1)E^2(z, w) + (1 - z_2\bar{w}_2)F^1(z, w),$$

we rearrange to yield

$$1 + z_1\bar{w}_1E^2(z, w) + z_2\bar{w}_2F^1(z, w) = \phi(z)\phi(w)^* + E^2(z, w) + F^1(z, w)$$

and write

$$E^2(z, w) = \sum_{j=1}^{d_1} E_j(z)E_j(w)^* \quad \text{and} \quad F^1(z, w) = \sum_{j=1}^{d_2} F_j(z)F_j(w)^*,$$

where $\{E_1, \dots, E_{d_1}\}$ is an orthonormal basis for $\mathcal{K}_\phi^2 \ominus Z_2\mathcal{K}_\phi$ and $\{F_1, \dots, F_{d_2}\}$ is an orthonormal basis for $\mathcal{K}_\phi^1 \ominus \mathcal{K}_\phi$.

It simplifies notation to write $E(z) = (E_1(z), \dots, E_{d_1}(z))$, $F(z) = (F_1(z), \dots, F_{d_2}(z))$. Then we have

$$E^2(z, w) = E(z)E(w)^*, F^1(z, w) = F(z)F(w)^*.$$

It can be shown that the map defined for each row vector $v \in \mathbb{C}^N$ and $z \in \mathbb{D}^2$ by:

$$(v, z_1vE(z), z_2vF(z))^t \mapsto (v\phi(z), vE(z), vF(z))^t$$

extends to a unitary U from $\mathbb{C}^N \oplus \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2}$ to itself. This is called a “lurking isometry argument.” Since this is a standard trick, we refer the reader to the proof of Lemma 6.7 in [22] where more details on this trick are provided.

We then obtain the formula

$$(12.3) \quad U \begin{bmatrix} I \\ z_1 E(z)^t \\ z_2 F(z)^t \end{bmatrix} = \begin{matrix} \mathbb{C}^N & \mathbb{C}^{|d|} \\ \mathbb{C}^{|d|} & \end{matrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} I \\ z_1 E(z)^t \\ z_2 F(z)^t \end{bmatrix} = \begin{bmatrix} \phi(z)^t \\ E(z)^t \\ F(z)^t \end{bmatrix}.$$

It is now possible to solve for $\phi(z)^t$ and see that it has a representation as in Corollary 1.11. Indeed,

$$\begin{aligned} A + Bd(z)(E(z), F(z))^t &= \phi(z)^t \\ C + Dd(z)(E(z), F(z))^t &= (E(z), F(z))^t \end{aligned}$$

and this implies

$$(12.4) \quad (E(z), F(z))^t = (I - Dd(z))^{-1}C$$

which implies

$$\phi(z)^t = A + Bd(z)(I - Dd(z))^{-1}C.$$

Of course, we could have applied the above argument to $\phi(z)^t$ to see that $\phi(z)$ has such a representation as well. (This annoyance stems from the fact that we prefer to have column-vector-valued spaces of functions.)

If we had a representation of ϕ using a *smaller* unitary U , say of size $(N + k_1 + k_2) \times (N + k_1 + k_2)$, then it is possible to reverse the arguments to get Agler kernels from equation (12.4) whose dimensions are (k_1, k_2) , which is not possible. \square

13. THEOREM 1.12, AN APPLICATION TO THREE VARIABLES

Theorem 1.12 can be slightly rephrased as follows:

Theorem 1.12. *If $p \in \mathbb{C}[z_1, z_2, z_3]$ has degree $(n, 1, 1)$ and no zeros on $\overline{\mathbb{D}^3}$, then*

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^3 (1 - |z_j|^2) SOS_j(z, z),$$

where SOS_2 and SOS_3 are sums of two squares, while SOS_1 is a sum of $2n$ squares.

Recounting all of the details of [25] would take us too far afield, so we shall only sketch the proof. It is mostly a matter of inserting Corollary 1.11 into the proper place in the proof.

Sketch of proof: Write $p(z_1, z_2, z_3) = a(z_1, z_2) + b(z_1, z_2)z_3$, where $a, b \in \mathbb{C}[z_1, z_2]$ have degree at most $(n, 1)$. Define

$$\tilde{a}(z_1, z_2) = z_1^n z_2 \overline{a(1/\bar{z}_1, 1/\bar{z}_2)}, \tilde{b}(z_1, z_2) = z_1^n z_2 \overline{b(1/\bar{z}_1, 1/\bar{z}_2)}.$$

Using the stability of p , it is possible to show a and $a + \tilde{b}$ have no zeros in \mathbb{D}^2 . It is shown in [25] that for $z_1, z_2 \in \mathbb{T}$ we may factor

$$|a(z_1, z_2)|^2 - |b(z_1, z_2)|^2$$

as a sum of two squares as follows:

$$|a(z_1, z_2)|^2 - |b(z_1, z_2)|^2 = \|E(z_1, z_2)\|^2 = |E_1(z_1, z_2)|^2 + |E_2(z_1, z_2)|^2,$$

where $E = (E_1, E_2)^t \in \mathbb{C}^2[z_1, z_2]$ is a (column) vector valued polynomial of degree at most $(n, 1)$. (The E here has no relation to the reproducing kernel E from earlier parts of the current paper. We are attempting to match the notation of [25].) Let

$$\tilde{E}(z_1, z_2) = z_1^n z_2 \overline{E(1/\bar{z}_1, 1/\bar{z}_2)}.$$

One can show

$$V := \frac{1}{a} \begin{bmatrix} \tilde{b} & \tilde{E}^t \\ E & \frac{E\tilde{E}^t - a(\tilde{a}+b)I}{a+\tilde{b}} \end{bmatrix}$$

is a 3×3 matrix valued inner function with the property that

$$(13.1) \quad V(z_1, z_2) \begin{bmatrix} p(z_1, z_2, z_3) \\ z_3 E(z_1, z_2) \end{bmatrix} = \begin{bmatrix} \tilde{p}(z_1, z_2, z_3) \\ E(z_1, z_2) \end{bmatrix}$$

for $(z_1, z_2, z_3) \in \mathbb{D}^3$. In order to use Theorem 1.8, we need to compute $\det V$. It is a direct calculation that

$$(13.2) \quad \det V = \frac{\tilde{a}(\tilde{a} + b)}{a(a + \tilde{b})},$$

which has degree at most $(2n, 2)$ since a, b have degree at most $(n, 1)$. A quick way to see this is to observe that

$$V \begin{bmatrix} a & b & 0 \\ 0 & E_1 & -\tilde{E}_2 \\ 0 & E_2 & \tilde{E}_1 \end{bmatrix} = \begin{bmatrix} \tilde{b} & \tilde{a} & 0 \\ E_1 & 0 & \frac{\tilde{a}+b}{a+\tilde{b}} \tilde{E}_2 \\ E_2 & 0 & -\frac{\tilde{a}+b}{a+\tilde{b}} \tilde{E}_1 \end{bmatrix}.$$

The first two columns on the right side are the result of (13.1), while the third column on the right comes from $(-\tilde{E}_2, \tilde{E}_1)\tilde{E} = 0$. If we now take the determinant of both sides, we get

$$(\det V)a(E_1\tilde{E}_1 + E_2\tilde{E}_2) = \tilde{a}\frac{\tilde{a}+b}{a+\tilde{b}}(E_1\tilde{E}_1 + E_2\tilde{E}_2),$$

which implies (13.2).

By Corollary 1.11 or equation (12.3) applied to $\phi = V^t$, there is a $(3 + 2n + 2) \times (3 + 2n + 2)$ unitary U such that

$$U \begin{bmatrix} I \\ z_1 G_1(z) \\ z_2 G_2(z) \end{bmatrix} = \begin{bmatrix} V(z) \\ G_1(z) \\ G_2(z) \end{bmatrix},$$

where G_1 is a $2n \times 3$ matrix valued rational function, and G_2 is a 2×3 matrix valued rational function.

If we multiply this equation on both sides by $Y = \begin{bmatrix} p \\ Z_3 E \end{bmatrix}$, we get via (13.1) that

$$U \begin{bmatrix} p \\ Z_3 E \\ Z_1 H_1 \\ Z_2 H_2 \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ E \\ H_1 \\ H_2 \end{bmatrix},$$

where $H_1 = G_1 Y$ and $H_2 = G_2 Y$. Since U is unitary, if we take norms (pointwise) of both sides and rearrange, we are left with

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1,2} (1 - |z_j|^2) \|H_j(z)\|^2 + (1 - |z_3|^2) \|E(z)\|^2.$$

Now, H_1 is $2n \times 1$ and H_2 is 2×1 , so that $\|H_1\|^2$ is a sum of $2n$ squares, $\|H_2\|^2$ is a sum of 2 and $\|E\|^2$ is a sum of 2. It is shown in [24] that the entries of H_1 and H_2 must be polynomials (specifically, see Claim 2 on page 351 of [24]). \square

14. FINAL COMMENTS

Let us end by highlighting further areas of work and a few questions in addition to the previously-asked Questions 1.13, 7.1, 10.2, and 10.5.

It would be interesting to completely describe the canonical spaces for more exotic inner functions or even the inner functions of Ahern [7], which are essentially rational in one of the variables.

For inner functions of more than two variables, certain decompositions of \mathcal{H}_ϕ analogous to Theorem 1.3 were discovered in [17] (and this was followed up in [23]), but it remains a challenge to form a useful decomposition of \mathcal{H}_ϕ involving “small” subspaces like $\mathcal{K}_\phi, \mathcal{K}_\phi^1, \mathcal{K}_\phi^2$. It would especially be interesting to decompose the reproducing kernel for \mathcal{H}_ϕ for rational inner ϕ using only finite dimensional spaces and their shifts. As already mentioned, an Agler decomposition as in Theorem 1.12 will not hold more generally, even for rational inner functions. Even when such a decomposition does hold, it not clear whether or not decompositions can be constructed naturally from orthogonal sums of

associated Hilbert spaces. The construction in the proof of Theorem 1.12 is most likely not of this form.

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